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NEUMANN TYPE BOUNDARY CONDITIONS FOR HAMILTON–JACOBI EQUATIONS

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Introduction. In this paper, we consider the classical first order Hamilton–Jacobi equations

$$H(x, u(x), Du(x)) = 0 \qquad \text{in } \Omega \tag{1}$$

where u is a scalar function on Ω bounded smooth open set of $\mathbf{R}^N$, where Du denotes the gradient of u and H—the Hamiltonian—is a given continuous function on $\bar{\Omega} \times \mathbf{R} \times \mathbf{R}^N$.

We want to study how it is possible to define for solutions of (1) Neumann type boundary conditions that is

$$\frac{\partial u}{\partial n} = 0 \qquad \text{on } \partial\Omega \tag{2}$$

where n is the unit outward normal to $\partial\Omega$. However, as it is remarked in P. L. Lions [25], A. Sayah [35], such a boundary condition is not always possible and has to be relaxed somehow.

Recently, M. G. Crandall and the author [8], [9] introduced a general notion of solutions of (1) (requiring only $u \in C(\Omega)$) and proved various properties of these solutions—called *viscosity solutions*—including stability and uniqueness (provided boundary conditions of Dirichlet type are imposed). This led to a complete treatment of (1) with, possibly, Dirichlet boundary conditions and we refer to M. G. Crandall, L. C. Evans and P. L. Lions [7]; P. L. Lions [26]; P. E. Souganidis [37]; G. Barles [3]; H. Ishii [22], [23]; M. G. Crandall and P. L. Lions [10], [11], [12], [13](*) . . . .

Our goal here is to adapt the notion of viscosity solutions of (1) in order to take into account boundary conditions of the form (2). Roughly speaking, we will present some weak formulation (in "viscosity style") of an equation combining (1) and (2) on $\partial\Omega$ and this will be interpreted as the relaxed form of (2). The precise definition is given in section I where we also motivate and explain this definition in the light of the so-called vanishing viscosity method which here consists of finding u_ϵ solution of the equation (3) below and letting ϵ go to 0_+

$$-\epsilon\Delta u_\epsilon + H_\epsilon(x, u_\epsilon, Du_\epsilon) = 0 \qquad \text{in } \Omega, \qquad \frac{\partial u_\epsilon}{\partial n} = 0 \qquad \text{on } \partial\Omega \tag{3}$$

where $H_\epsilon \to H$ as $\epsilon \to 0_+$ (one can take $H_\epsilon = H$ as well).

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(*) The reader should be aware that this list is by no means complete!