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MAXIMAL FUNCTIONS FOR CONVEX CURVES

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§1. Introduction. Let $\Gamma:\mathbf{R}\to\mathbf{R}^n$ be a continuous curve in $\mathbf{R}^n$, $n \geqslant 2$, with $\Gamma(0)=0$. To Γ we associate the Hilbert transform operator $\mathfrak{H}$ and maximal operator $\mathfrak{M}$ defined by

$$\mathfrak{H}f(x) = \text{p.v.}\int_{-\infty}^{\infty} f(x-\Gamma(t))t^{-1}\,dt \tag{1}$$

$$\mathfrak{M}f(x) = \sup_{r>0} r^{-1}\int_0^r |f(x-\Gamma(t))|\,dt \tag{2}$$

where $x \in \mathbf{R}^n$ and f is in an appropriate class of functions on $\mathbf{R}^n$. For which curves Γ and which indices p are these operators bounded on $L^p(\mathbf{R}^n)$? Since the appearance of $\mathfrak{H}$ in the literature twenty years ago, and especially in recent years, these problems have attracted a great deal of interest. See [SW] for a survey of results through 1977. Further developments are found in [NSW2], [Ne], [Wn], [NVWW1], [NVWW2], [CaW], [Ch], and [CNVWW].

In this paper, we will consider *plane* curves ($n=2$) Γ with

$$\Gamma(t) = (t,\gamma(t)) \qquad \text{where} \quad \gamma:\mathbf{R}\to\mathbf{R} \text{ is convex for } t \geqslant 0, \text{ continuous}$$
$$\text{and satisfies } \gamma(0)=\gamma'(0)^+=0. \tag{3}$$

(The convexity hypothesis means, of course, that $[\gamma(C)-\gamma(B)]/(C-B) \geqslant [\gamma(B)-\gamma(A)]/(B-A)$ for $0 \leqslant A < B < C$.)

For curves satisfying (3) with γ odd, a condition *equivalent* to the boundedness of $\mathfrak{H}$ on $L^2(\mathbf{R}^2)$ is known (see [NVWW1]). We will show here that the same condition is sufficient for $\mathfrak{M}$ also to be bounded on $L^2(\mathbf{R}^2)$. This condition is not necessary, however (see [CaW]). An interesting feature of the proof is the use of certain "balls" appropriate to the geometry of Γ, specifically parallelograms which are tangent to Γ.

§2. Statement of theorem and sketch of proof. Let Γ be a plane curve satisfying (3). Then, γ has one-sided derivatives for every $t>0$ and a two-sided derivative except on a countable set. Let us write $\gamma'(t)$ for the right-hand

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