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# ON EXTENDING SOLUTIONS TO DIRICHLET PROBLEMS ACROSS THE BOUNDARY AS SOLUTIONS

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**1. Introduction.** On a $C^\infty$ manifold $M$ with boundary, if $u$ is an extendible distribution satisfying

$$Pu \in C^\infty(M), \qquad u|_{\partial M} \in C^\infty(\partial M), \tag{1.1}$$

it is natural to ask when $u$ can be extended across $\partial M$ as a solution, that is, to $\tilde{u} \in \mathscr{D}'(\tilde{M})$ such that $P\tilde{u} \in C^\infty(\tilde{M})$, for some open manifold $\tilde{M}$ extending $M$ across $\partial M$. Here suppose that $P$ is a second order differential operator on $\tilde{M}$, noncharacteristic with respect to $\partial M$, with real principal symbol $p$ having fiber-simple characteristics:

$$d_{\text{fiber}}p \neq 0 \qquad \text{on } p^{-1}(0) \cap (T^*\tilde{M}\backslash 0). \tag{1.2}$$

When $\partial M$ is everywhere elliptic or everywhere hyperbolic with respect to $P$, it is classical that an extension $\tilde{u}$ satisfying $P\tilde{u} \in C^\infty(\tilde{M})$ can be found. However, when null bicharacteristics tangent to $\partial T^*M$ are present, extensions as solutions do not always exist. Before discussing this case, we recall the classification of boundary points.

Let $\iota : \partial M \to M$ be the inclusion inducing the projection $\iota^* : \partial T^*M \to T^*\partial M$. Then the elliptic, hyperbolic, and glancing regions are

$$\begin{aligned} E &= \{\sigma \in T^*\partial M\backslash 0 : p \neq 0 \text{ on the line } \iota^{*-1}(\sigma)\} \\ H &= \{\sigma \in T^*\partial M\backslash 0 : p \text{ has (two) simple zeroes in } \iota^{*-1}(\sigma)\} \\ G &= \{\sigma \in T^*\partial M\backslash 0 : p \text{ has a double zero in } \iota^{*-1}(\sigma)\}. \end{aligned} \tag{1.3}$$

Let $x$ be a real $C^\infty$ function on $\tilde{M}$ vanishing simply on $\partial M$ with $x > 0$ in $\mathring{M}$ near $\partial M$. Then if $\sigma = \iota^*\rho \in G$, where $\rho$ is the double zero, we have $\{p, x\}(\rho) = 0$ ($\{\ ,\ \}$ is the Poisson bracket). We can write $G = G_d \cup G_g \cup G_0$ where

$$\begin{aligned} G_d &= \{\sigma = \iota^*\rho \in G : p(\rho) = 0 \text{ and } \{p, \{p,x\}\}(\rho) > 0\} \\ G_g &= \{\sigma = \iota^*\rho \in G : p(\rho) = 0 \text{ and } \{p, \{p,x\}\}(\rho) < 0\} \\ G_0 &= \{\sigma = \iota^*\rho \in G : p(\rho) = 0 \text{ and } \{p, \{p,x\}\}(\rho) = 0\} \end{aligned} \tag{1.4}$$

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