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CHARACTERIZATION OF SIMPLE CLOSED GEODESICS ON FRICKE SURFACES

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This paper characterizes in a recursively enumerable way the simple closed geodesics on any Riemann surface of signature $(0;2,2,2,\infty)$. We also show that this same characterization is valid for any surface of signature $(1;\infty)$ or $(0;\infty,\infty,\infty,\infty)$, each of which is covered by a $(0;2,2,2,\infty)$ surface.[1]

We shall think of the $(0;2,2,2,\infty)$ surface as being given by the quotient of the upper half plane H by a Fuchsian group Φ. It is well known then that closed geodesics (with nontrivial homotopy classes) correspond to conjugacy classes of hyperbolic elements of Φ. The axis of any element in a given class projects to the same closed geodesic on the surface (see Beardon [B], p. 173). Our algorithm, then, will recursively enumerate exactly those (conjugacy classes of) hyperbolic elements whose axes project to *simple* closed geodesics.

In the case where Φ (the $(0;2,2,2,\infty)$ group) is $\Gamma^3 = \langle A = B^3 \mid B \in \mathrm{SL}(2,Z) = \Gamma(1)\rangle$, i.e., the group generated by cubes of elements of the modular group, the recursive enumeration is equivalent to the construction of the discrete portion of the venerable Markov spectrum of diophantine approximation. More specifically, Hurwitz's theorem states that for any irrational θ, $|\theta < p/q| < 1/(\sqrt{5}\,q^2)$ has an infinity of solutions. $\mu_1 = \sqrt{5}$ is best possible here as $\theta_1 = (1+\sqrt{5})/2$, the golden mean, shows. However, if we eliminate θ_1 (and its $\Gamma(1)$-equivalents) then we can replace μ_1 by $\mu_2 = \sqrt{8}$ which is best possible as $\theta_2 = \sqrt{2}$ shows. If next we eliminate the $\Gamma(1)$-equivalents of θ_2, we may replace μ_2 by $\mu_3 = \sqrt{221}/5$ which is best possible as $\theta_3 = (9+\sqrt{221})/10$ shows. Continuing in this way gives us a sequence $\mu_i \uparrow 3$ (called the (discrete) Markov spectrum) and an associated sequence of ($\Gamma(1)$-equivalence classes of) quadratic irrationals θ_i, called the Markov Quadratic Irrationalities (MQI). These can be recursively enumerated (Koksma [K] p. 29; Cassels [C], p. 18; and §2 of the present paper).

For Γ^3 our result is simply stated.

THEOREM 1 (Γ^3). *γ is simple on $\Gamma^3 \Leftrightarrow$ the fixed points of γ, ζ_γ and ζ'_γ, are in MQI.*

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