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SUPPLEMENT TO "VARIATION OF MIXED HODGE STRUCTURE ARISING FROM FAMILY OF LOGARITHMIC DEFORMATIONS II: CLASSIFYING SPACE"

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In this note, we will define a graded polarization (abbreviated as GP) of the mixed Hodge structure (abbreviated as MHS) on $H^n(X - Y, \mathbf{Q})$, where X is a smooth projective variety over $\mathbf{C}$ and Y is a (reduced) normal crossing divisor (abbreviated as NCD) on X, and give some supplements to [U.2]. This note is based on a small meeting of the three authors at RIMS, 1–6 Oct. '84.

1. Graded polarization on $H^n(X - Y, \mathbf{Q})$. Let X and Y be as above and set $r := \dim X$. Locally on X, Y is a union of its irreducible components; $Y = \bigcup_{i \in I} Y_i$. Set $Y_j := \bigcap_{j \in J} Y_j$ for a subset $J \subset I$, $\tilde{Y}^s := \coprod_J Y_J$ where J runs the subsets of I with $\# J = s\ (\geqslant 1)$ and $\tilde{Y}^0 = X$. These $\tilde{Y}^s$ patch together and have global meaning. (cf. II (3.1.4) in [D.1])

Next, choose a polarization on X $\omega \in H^2(X, \mathbf{Z}) \cap H^{1,1}(X)$ (i.e., the cohomology class of a very ample invertible sheaf). Then ω induces a polarization ω_J of Y_J and also a polarization of $\tilde{Y}^s$, $\omega_s := \bigoplus_{\# J = s} \omega_J \in H^2(\tilde{Y}^s, \epsilon^s_{\mathbf{Z}}) \cap H^{1,1}(\tilde{Y}^s)$, where $\epsilon^s_{\mathbf{Z}}$ denotes the sheaf of orientations on $\tilde{Y}^s$. (cf. ibid.)

These data define a polarization of HS on $H^m(\tilde{Y}^s, \epsilon^s_{\mathbf{Q}})$ by the following well-known procedure in Kähler geometry. Let $\nu_J : H^{2(r-s)}(Y_J, \mathbf{Z}) \tilde{\to} \mathbf{Z}$ be the trace map with $\nu_J(\omega_J^{r-s}) = 1$ ($\# J = s$) and set $\nu_s := \sum_{\# J = s} \nu_J : H^{2(r-s)}(\tilde{Y}^s, \epsilon^s_{\mathbf{Z}}) \to \mathbf{Z}$. Let L denote the operator of "cup-product with ω_s" on $H^*(\tilde{Y}^s, \epsilon^s_{\mathbf{Z}})$ and define

$$P^m(\tilde{Y}^s, \epsilon^s_{\mathbf{Q}}) := \operatorname{Ker}\left(L^{(r-s)-m+1} : H^m(\tilde{Y}^s, \epsilon^s_{\mathbf{Q}}) \to H^{2(r-s)-m+2}(\tilde{Y}^s, \epsilon^s_{\mathbf{Q}})\right)$$

(the primitive part). Then by using the Lefschetz decomposition

$$H^m(\tilde{Y}^s, \epsilon^s_{\mathbf{Q}}) = \bigoplus_{a \geqslant 0} L^a P^{m-2a}(\tilde{Y}^s, \epsilon^s_{\mathbf{Q}}) \tag{1.1}$$

we can define a polarization Q'_s of HS on $H^m(\tilde{Y}^s, \epsilon^s_{\mathbf{Q}})$ by the formula

$$Q'_s(u, v) := \sum_a (-1)^{(m-2a)(m-2a+1)/2} \nu_s\left(u_{m-2a} \cup v_{m-2a} \cup \omega_s^{(r-s)-(m-2a)}\right) \tag{1.2}$$

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