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ON THE DIFFERENTIABILITY OF $\mathbf{O}(n)$ INVARIANT FUNCTIONS OF SYMMETRIC MATRICES

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0. Introduction. Consider a real valued function F defined on the linear space, $S^{n\times n}$, of $n \times n$ symmetric matrices. We shall denote by f the restriction of F to the diagonal matrices. If F is r times continuously differentiable, then so is f; and if F is invariant under the action of the orthogonal group, $\mathbf{O}(n)$, on $S^{n\times n}$ $((O,S)\to O^TSO)$ then f is invariant under the action of that subgroup which preserves diagonal matrices (i.e., the group S_n of permutations of the diagonal elements).

The purpose of this paper is to prove the converse: that if f is both S_n invariant and r times continuously differentiable, then F, defined to be invariant under the $\mathbf{O}(n)$ action, is also r times continuously differentiable.

This was conjectured by Ball [1984], where he proved the corresponding result for $C^{k,\alpha}$, functions whose derivatives satisfy Hölder estimates, by constructing certain invariant elliptic operators. For C^∞, the result follows from Glaeser [1963].

The basis of our proof is the Lipschitz continuity of the map from symmetric matrices to their (ordered) eigenvalues, and estimates on the derivatives of this map, when restricted to those matrices with distinct eigenvalues. These are used to verify that the Taylor polynomial for F, constructed in §2, satisfies error estimates which allow us to conclude that F has r continuous derivatives.

In §2, we construct the Taylor polynomial for F at all points, not just at those with distinct eigenvalues. We could have avoided this difficulty. The set of symmetric matrices with multiple eigenvalues is closed and sparse, hence functions r times continuously differentiable off that set whose derivatives have limits there are r times continuously differentiable on that set as well (see Ball [1984]). We hope, however, that the construction will aid in understanding the result.

In §1 we introduce some notation. The main theorem is proved in §2, and the main estimates relegated to §3.

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1. Notation

Calculus. Let $C^r_{\mathbf{O}(n)}(S^{n\times n})$ denote the r times continuously differentiable functions on $S^{n\times n}$ which are invariant under conjugation by elements of the

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