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THE DIMENSION OF SMOOTHING COMPONENTS

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Introduction. As is well known, any complex-analytic germ (X,x) with isolated singularity admits a miniversal deformation

$$\begin{array}{ccc} (X,x) & \xhookrightarrow{i} & (\mathscr{Y},y) \\ & & \downarrow F \\ & & (S,s) \end{array}$$

(cf. Grauert [Gr] and Bingener [Bi]). The Zariski tangent space of the base T_sS is naturally isomorphic to the module of first order deformations $T^1_{X,x}$ and the latter admits a simple description so that its length ($=\mathbf{C}$-dimension) is easy to calculate. If (X,x) has no obstructed deformations (e.g., (X,x) is a complete intersection), then (S,s) is nonsingular and so a good formula for $\dim(S,s) = \dim_{\mathbf{C}} T_sS$ is available. In general however, the germ (S,s) is not smooth and indeed, it may have irreducible components of various dimensions. (Perhaps the simplest example of this phenomenon (due to Pinkham [Pi]) is the affine cone over a smooth rational curve of degree 4 in $\mathbf{P}^4$). This confronts us with the problem of computing the dimensions of these components. At first sight this seems an impossible task, so it is already amazing that there is a simple conjectural formula for the dimension of a smoothing component (S',s) of (S,s), due to Wahl [Wa]. Recall that an irreducible component of (S,s) is called a *smoothing component* if the general fibre of F over this component is smooth. Denote by Δ the complex unit disc and let $j:(\Delta,0)\to(S',s)$ be a morphism such that the pull-back $f=F^*(j):(\mathscr{X},x)\to(\Delta,0)$ has smooth general fibre.

Conjecture of Wahl. $\dim(S',s)=\dim_{\mathbf{C}}\mathrm{Coker}(\Theta_{\mathscr{X}/\Delta,x}\to\Theta_{X,x})$. The interest of the conjecture is that for a given f the RHS is in general easier to determine than the LHS, although this may still be hard. Wahl himself verified his conjecture in a number of cases: when $T^2_{X,x}=0$ or when (X,x) can be globalized in a suitable manner [Wa].

The main purpose of this note is to prove this conjecture. As expected by Wahl, the proof works in a general deformation-theoretic setting so that for instance an analogous statement holds for the smoothing components of a compact complex-analytic space. We briefly indicate (in §3) how the arguments are made to work in this general context and we spell the result out for a few concrete cases.

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