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LIE GROUPS AND TWISTOR SPACES

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§1. Introduction. The relationship between harmonic maps $f: S \to M$ of surfaces into Riemannian manifolds and complex manifold theory has received much attention in recent years. Beginning in 1967, when Calabi [6] showed how one could classify the harmonic maps $f: S^2 \to S^k$ in terms of holomorphic data, much emphasis has been placed on the problem of classifying the harmonic maps $f: S^2 \to M$ for more general Riemannian symmetric spaces, notably $M = \mathbf{CP}^n$ and, more generally $M = G_k(\mathbf{C}^{k+n})$. For excellent reports on this subject, see [11], [12], and [15].

Inspired by the Penrose program, Rawnsley [22] and others considered the twistor bundle $\tilde{\mathscr{T}}(M) \to M$ where the fiber $\tilde{\mathscr{T}}_x(M)$ for $x \in M$ consists of the space of complex structures $\mathscr{J}: T_xM \to T_xM$, $\mathscr{J}^2 = -id$. When M is equipped with an affine connection, ∇, $\tilde{\mathscr{T}}(M)$ inherits an almost complex structure and a complex plane field $\mathscr{H}^\nabla$ on $\tilde{\mathscr{T}}(M)$ transverse to the fibers of $\tilde{\mathscr{T}}(M) \to M$. This almost complex structure and plane field has the property that every $\tilde{f}: S \to \mathscr{T}(M)$ where S is a Riemann surface, $\tilde{f}_*$ is complex linear, and $\tilde{f}$ is tangent to $\mathscr{H}^\nabla$ projects to M to yield a harmonic map $f: S \to M$. (It is *not* true, in general, that *every* harmonic f is the projection of such an $\tilde{f}$.) This gives a relationship between almost complex geometry and harmonic maps.

Unfortunately, "almost complex" is still far from "complex". It is not much easier to construct solutions to "almost complex" equations than to construct solutions to the harmonic equations directly. The method above has been most successful when either $\tilde{\mathscr{T}}(M)$ actually turns out to be complex (a quite rare occurrence) or else some subspace of $\tilde{\mathscr{T}}(M)$ inherits an actual complex structure.

In this paper, we concentrate entirely on the case when M is a simply connected Riemannian symmetric space, ∇ is the Levi–Civita connection and we only consider the *metric* twistor bundle $\mathscr{T}(M) \subseteq \tilde{\mathscr{T}}(M)$ whose fibers $\mathscr{T}_x(M)$ for $x \in M$ consist of the *orthogonal* complex structures. Since $\mathscr{T}(M)$ is tangent to $\mathscr{H}$, we may formulate many of the same constructions for $\mathscr{T}(M)$ as have been made for $\tilde{\mathscr{T}}(M)$.

However, except when M is a Riemann surface or a space of constant sectional curvature, $\mathscr{T}(M)$ is not a complex manifold and $\mathscr{H}$ is not a holomorphic plane field on $\mathscr{T}(M)$. After some preliminary work in §1 (mainly to fix notation), we remedy this situation in §2 by the most naive possible means:

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