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ON THE SPECTRUM OF THE HECKE GROUPS

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§1. Introduction. Let $\mu \geqslant 1$ and Γ_μ be the subgroup of PSL(2, R) generated by the transformations S and T_μ where

$$S(z) = -1/z, \qquad T_\mu(z) = z + 2\mu.$$

Here $z \in \mathscr{H} = \{z \in \mathbf{C} : \operatorname{im}(z) > 0\}$. The limit set Λ_μ of Γ_μ is defined to be the set of limit points of any fixed orbit of Γ_μ. Since Γ_μ is a discrete subgroup of SL(2, R) the set $\Lambda_\mu \subset \mathbf{R} \cup \{\infty\}$. For $\mu = 1$ it is easy to see that $\Lambda_1 = \mathbf{R}$, while for $\mu > 1$ it is a Cantor-like subset of $\mathbf{R} \cup \{\infty\}$. We denote by $\delta(\mu)$ the Hausdorff dimension of Λ_μ. The function $\mu \to \delta(\mu)$ has been studied by a number of different authors [B, P1, PI, P-S] and it is known from these papers that $\delta(1) = 1$, $\delta(\infty) \equiv \lim_{\mu\to\infty}\delta(\mu) = 1/2$ and δ is a strictly decreasing Lipschitz continuous function on $[1, \infty)$.

In this note we find it more advantageous to work with the base eigenvalue $\lambda(\mu)$ of the Laplacian over the fundamental domain of $\Gamma(\mu)$. $\lambda(\mu)$ is related to $\delta(\mu)$ by

$$\lambda = \delta(1 - \delta). \tag{1}$$

Clearly then λ is strictly increasing, beginning at 0 when $\mu = 1$ and tending to $1/4$ as $\mu \to \infty$. In Figure 1 we have graphed the functions $\mu \to \lambda(\mu)$ and $\mu \to \delta(\mu)$. The graph is taken from the numerical data listed in Table 2 which was obtained by the same method that we employed in [P-S] for Schottky groups. From the graph it appears that $\lambda(\mu)$ is convex down and the main result of this note is an analytic proof of this fact:

THEOREM 1. *For μ in the range $(1, \infty)$, $\lambda(\mu)$ is analytic in μ and*

$$\frac{d^2\lambda}{d\mu^2} < 0.$$

§2. Proof of results. Our proof of Theorem 1 is based on the theory of partial differential equations and concepts associated with this theory (see [P1, P-S]).

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