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HEAT EQUATION AND COMPACTIFICATIONS OF COMPLETE RIEMANNIAN MANIFOLDS

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1. Introduction. Let M be a complete Riemannian manifold whose Ricci curvature is bounded from below. A compactification $\overline{M}$ of M is a compact Hausdorff space which contains M as a dense subspace. We assume that $\overline{M}$ is first countable [6, p. 186]. In particular, for each $\bar{x} \in \overline{M}$ there exists a sequence $x_n \in M$ with $x_n \to \bar{x}$, where the arrow denotes convergence. If $f \in C(\overline{M})$, the space of continuous functions on $\overline{M}$, then f is uniquely determined by its restriction to M. The symbol $\overline{M}_\infty$ will denote the complement of M in $\overline{M}$.

Given any $f \in C(\overline{M})$, one looks for functions $f(x,t) \in C(\overline{M} \times [0,\infty))$ satisfying the following three conditions:

$$\text{(i)} \qquad \left(\frac{\partial}{\partial t} - \Delta\right) f(x,t) = 0 \qquad (x,t) \in M \times (0,\infty)$$

$$\text{(ii)} \qquad f(x,0) = f(x) \qquad x \in M \tag{1.1}$$

$$\text{(iii)} \qquad f(x,t) = f(x) \qquad (x,t) \in \overline{M}_\infty \times [0,\infty).$$

Here Δ is the Laplacian associated to the Riemannian metric of M. Moreover, it is assumed that $f(x,t)$ will be twice continuously differentiable in x and once continuously differentiable in t, for $(x,t) \in M \times (0,\infty)$. The parabolic problem (1.1) is overdetermined. In fact, since the Ricci curvature of M is bounded from below, one has uniqueness for the heat equation problem in the space of bounded continuous functions [4], [13]. This implies that $f(x,t)$, for $(x,t) \in M \times (0,\infty)$, is completely determined by the first two conditions of (1.1). There is at most one continuous extension to $\overline{M} \times (0,\infty)$. A priori, it may be impossible to prescribe the values $f(x,t)$, for $(x,t) \in \overline{M}_\infty \times [0,\infty)$, as required by (iii).

The main purpose of this paper is to give a simple geometric criterion for the solvability of (1.1). If $x \in M$ and $\gamma > 0$, then $B(x,\gamma)$ will denote the geodesic ball of radius γ, centered at x. One defines the following:

(Ball Convergence Criterion) If $x_n \in M$ is a sequence with $x_n \to \bar{x} \in \overline{M}$, then for all $\gamma > 0$, $B(x_n,\gamma) \to \bar{x}$. (1.2)

Equivalently, one may require convergence for some $\gamma > 0$. Our main theorem is as follows:

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