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THREE NOTES ON $R^{(1)}$

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1. Introduction. It is a classical result (and sometimes part of the definition—see [1, p. 480], [5, pp. 293–294], [7, pp. 115–116], and [15, p. 82]) that if R is a Krull domain and $\mathscr{H} = \{p \in \operatorname{Spec} R;\ \text{height}\ p = 1\}$, then $R = \bigcap \{R_p;\ p \in \mathscr{H}\}$ and if $q \in \mathscr{H}$ and $\mathscr{H}' = \mathscr{H} - \{q\}$, then $R \subset \bigcap \{R_p;\ p \in \mathscr{H}'\}$. In [4, (7.2.3)], Grothendieck proved the following related result: If R is a local domain, then $R^{(1)} = \bigcap \{R_p;\ p \in \mathscr{H}\}$ is a finite R-module if and only if the following condition holds:

If $z \subset p^*$ are prime ideals in the completion R^* of R such that $z \in \operatorname{Ass} R^*$ and height $p^* \cap R > 1$, then height $p^*/z > 1$. (1.1)

(Note that if R is integrally closed here, then $R^{(1)}$ being a finite R-module is equivalent to $R = R^{(1)}$, so this result can be considered a form of a generalization of the classical result.)

Another generalization of the classical result is: If R is a Noetherian domain and $\mathscr{P} = \{p \in \operatorname{Spec} R;\ p$ is a prime divisor of a nonzero principal ideal$\}$, then $R = \bigcap \{R_p;\ p \in \mathscr{P}'\}$. (Concerning this, see the parenthetical statement at the end of (2.3).)

In this paper we prove another such generalization: $R^e = \bigcap \{R_p;\ p$ is essential$\}$ is always a finite R-module, when R is a semi-local domain, and if q is a maximal essential prime ideal in R, then $R \subset \bigcap \{R_p;\ p$ is essential and $p \neq q\}$. This is proved in (3.1), after proving in Section 2 some characterizations of and facts concerning essential prime ideals, and in (3.1) we also characterize in terms of $\mathscr{S}$ when $\bigcap \{R_p \mid p \in \mathscr{S}\}$ is a finite R-module: namely, every essential prime ideal is contained in some $p \in \mathscr{S}$. Two corollaries are: (1) If R is an unmixed semi-local domain, then $R^{(1)}$ is a finite R-module; and, (2) If R is a semi-local domain, then $R^{(w)} = \bigcap \{R_p;\ p \in \operatorname{Spec} R$ and p is not maximal$\}$ is a finite R-module if and only if no maximal ideal in R is an essential prime ideal. Section 3 also contains a characterization of R^e in terms of prime divisors of principal ideals in over-rings, where R is either a semi-local domain or a Noetherian domain with finite integral closure. Then, in Section 4, a new proof is given for Grothendieck's result mentioned above. Finally, Section 5 contains several characterizations of when A^a (see (2.3)) is a finite A-module for all finite integral extension domains A of a given semi-local domain R, and it also contains a

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