Vol. 51, No. 1 DUKE MATHEMATICAL JOURNAL © March 1984

THE MODULI OF CURVES IS STABLY RATIONAL FOR $g \leqslant 6$

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It is well known that the moduli space of curves is unirational for $g \leqslant 10$. (cf. [11]). Our aim is to prove that a stronger result holds for $g \leqslant 6$.

Definition (see e.g. [13]). An algebraic variety X is called stably rational if $X \times \mathbf{P}^k$ is birational to $\mathbf{P}^n$ for some k, n. If everything here is defined over a field F then we say that X is stably rational over F. It seems to be unknown whether stably rational varieties are rational or not. This is a special case of a problem of Zariski (cf. [7]).

Our aim is to prove the following:

THEOREM. *Let F be any algebraically closed field and $\mathscr{M}_g(F)$ the moduli space of curves of genus g over F. Then $\mathscr{M}_g(F)$ is stably rational (over F) for $g \leqslant 6$.*

Proof. For $g = 1,2$ it is even rational, the latter case by Igusa [5].

For $g = 3,4,5$ the proof is easy and probably well known; we shall sketch it. Let $\mathscr{H}_g(F)$ be the Hilbert scheme of canonically embedded curves of genus g. Then $\mathscr{H}_g(F)$ is birationally a $\mathrm{PGL}(g, F)$ bundle over $\mathscr{M}_g(F)$, and it is locally trivial since the canonical bundle of a curve is defined over the field of definition of the curve.

Now for $g = 3,4,5$ the general canonical curves are known to be complete intersections, so the rationality of $\mathscr{H}_g(F)$ is easy.

The main part of the proof is the case $g = 6$.

For a curve C let $F[C]$ denote its field of definition over F, equivalently, the residue field of $[C] \in \mathscr{M}_g(F)$. The proof rests on the following observation:

PROPOSITION. *For general C the canonical image (denoted by C') lies on a unique del Pezzo surface D_C of degree 5. Moreover D_C is defined over $F[C]$.*

Proof. This fact was essentially known to Noether [8], cf. also [9] [1]. It was pointed out to us by Ron Donagi and Joe Harris that the unicity of D_C can be established by direct geometric arguments, so D_C is defined over an inseparable extension of $F[C]$. The inseparability can be dealt with by various methods. We prefer the following one which is not the shortest but gives an explicit way to write down the equations of D_C. We shall consider the projective resolution of the homogeneous coordinate ring S_C of C'. Let S be the homogeneous coordinate ring of $\mathbf{P}^5$ with its natural grading.

Received May 30, 1983.