Vol. 51, No. 1 DUKE MATHEMATICAL JOURNAL © March 1984

A SIMPLE CRITERION FOR LOCAL HYPERSURFACES TO BE ALGEBRAIC

JAY A. WOOD

Introduction. In this paper we give a necessary and sufficient condition for d pieces of hypersurface to be contained in an algebraic hypersurface of degree d.

Given d pieces of hypersurface $\gamma_1, \ldots, \gamma_d$ in $(n+1)$-dimensional projective space P^{n+1}, suppose there is a line L_0 in P^{n+1} which intersects each of the γ_i transversely. Fix affine coordinates $(x_0, \ldots, x_n)$ on P^{n+1}, and fix line coordinates $(m_1, \ldots, m_n, b_1, \ldots, b_n)$ (i.e., local coordinates on $\mathrm{Gr}(1, n+1)$, the Grassmannian of all lines in P^{n+1}), where a line L is given by $x_k = m_k x_0 + b_k$, $k = 1, \ldots, n$. It can be assumed that L_0 has line coordinates $m_k = 0$, $b_k = 0$, for all k. For convenience, write $m = (m_1, \ldots, m_n)$, $b = (b_1, \ldots, b_n)$.

A line $L = L(m,b)$ near L_0 will intersect each γ_i in a point $P_i = P_i(m,b)$. Let $X_i = X_i(m,b)$ be the Oth coordinate of P_i. We can now state the main result.

THEOREM. *There exists an algebraic hypersurface γ of degree d containing each γ_i, $i = 1, \ldots, d$, if and only if*

$$\sum_i (\partial^2 X_i)/(\partial b_k \partial b_1) = 0,$$

for all $k, 1 = 1, \ldots, n$, the summation running over $i = 1, \ldots, d$.

Results of this kind have been known earlier. In fact, the theorem above is just the main theorem of the author's dissertation [W], restated in terms of the specific choice of coordinates given above. However, the proof given below is completely different from the proof in [W]. In particular, the present proof is shorter and less computational.

In addition, the theory above generalizes to arbitrary dimensions and degrees a theorem due to Lie [L] and Scheffers [S] for four curves in the plane ($d = 4$, $n = 1$). The method of proof is essentially the same, only recast in a different coordinate system. We believe Scheffers could also have provided a proof of the present theorem.

Necessity. Suppose there exists an algebraic hypersurface γ of degree d, which contains each γ_i, and which satisfies the degree d polynomial equation

$$p(x_0, \ldots, x_n) = 0.$$

Received July 6, 1983.