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PRIME DIVISORS OF FOURIER COEFFICIENTS OF MODULAR FORMS

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§1. Introduction. The Ramanujan τ-function is defined by

$$\Delta = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n.$$

Ramanujan [6] investigated the divisibility properties of $\tau(n)$ and conjectured that $\tau(n) \equiv 0 \pmod{691}$ for almost all n. This was verified by Watson [12]. Serre [9] has strengthened this to the following assertion: given an integer d, we have $\tau(n) \equiv 0 \pmod{d}$ for almost all n (i.e., for all n excepting a set of density 0). In fact, Serre's result holds for the Fourier coefficients of modular forms of integral weight for any congruence subgroup of $SL_2(\mathbf{Z})$.

The purpose of this paper is to further investigate the divisibility properties of these coefficients. For definiteness, we shall state the results for τ, though they apply to more general multiplicative functions.

We first prove the following strengthening of Serre's result: given d as above, $\tau(n)$ is divisible by d^{ω}, where $\omega = [\delta \log\log n]$, for almost all n. (Here δ is a positive constant depending on d.) We then consider the effect of varying d. Denote by $\nu(n)$ the number of *distinct* prime divisors of n. Assuming the Generalized Riemann Hypothesis (GRH), we show that

$$\sum_{\substack{p \leqslant x \\ \tau(p) \neq 0}} (\nu(\tau(p)) - \log\log p)^2 \ll \pi(x) \log\log x$$

and

$$\sum_{\substack{n \leqslant x \\ \tau(n) \neq 0}} \left(\nu(\tau(n)) - \frac{1}{2}(\log\log n)^2\right)^2 \ll x(\log\log x)^3 \log_4 x.$$

(Here, $\log_4 x = \log\log\log\log x$.) In particular, given $\epsilon > 0$, we have

$$|\nu(\tau(p)) - \log\log p| < (\log\log p)^{1/2+\epsilon}$$

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