Vol. 50, No. 4 DUKE MATHEMATICAL JOURNAL © December 1983

CATEGORY $\mathscr{O}'$, $\mathfrak{n}$-HOMOLOGY AND THE REDUCIBILITY OF GENERALIZED PRINCIPAL SERIES REPRESENTATIONS

DAVID H. COLLINGWOOD

1. Introduction. Let G be a connected semisimple real matrix group, $\mathfrak{g}$ the complexified Lie algebra of G and $\mathfrak{U}(\mathfrak{g})$ the corresponding enveloping algebra. Investigating the representation theory of G, typically, leads one to work within one of the following two categories: Category $\mathscr{H}$, consisting of Harish-Chandra modules for G—a "global category"; Category $\mathscr{O}'$, consisting of finite length $\mathfrak{U}(\mathfrak{g})$-modules satisfying a certain finiteness condition—an "infinitesimal category". Each category contains a collection of *standard modules*. In category $\mathscr{H}$, these are the *generalized principal series representations*; in category $\mathscr{O}'$, the *Verma modules*. In either setting, the standard modules have finite length, making it a natural problem to study the structure of their Jordan–Hölder series; the so-called *composition series problem*. As a first step, one seeks necessary and sufficient conditions for the reducibility of standard modules. For Verma modules, this criterion (and much more) is contained in the celebrated Bernstein–Gelfand–Gelfand theorem [7]. For generalized principal series representations, reducibility conditions were obtained by B. Speh and D. Vogan [20]. On a very formal level, these two theorems look strikingly similar. Because of this, and the existence of a "nice" functor between the categories in question, it is instinctive to try and connect these two results. Using the Bernstein–Gelfand–Gelfand theorem, the Jacquet functor and elegant connections between the asymptotic behavior of matrix coefficients, characters, $\mathfrak{n}$-homology and intertwining operators, we offer a new proof of the Speh–Vogan theorem. In a nutshell, given a generalized principal series representation π, we obtain adequate control over the possible asymptotic exponents of π, to detect and exhibit reducibility. In particular, one type of reducibility has an especially pleasing explanation in category $\mathscr{O}'$.

Let G be as above, $P = MAN$ the Langlands decomposition of a cuspidal parabolic subgroup of G, δ a discrete series representation for M and e^{ν} a (non-unitary) character of the vector group A. We call the induced module $\pi(\delta,\nu) = I_P^G(\delta \otimes e^{\nu})$ (normalized induction) a *generalized principal series representation*. Our main result is

(1.1) THEOREM (Speh–Vogan [20]). *Let G be a connected semisimple real matrix group and $\pi(\delta,\nu)$ a generalized principal series representation. Fix a compact Cartan subgroup $T \subseteq M$ (which exists by the cuspidality of P) and let*

Received May 9, 1983.