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SURFACES WITH A HYPERELLIPTIC HYPERPLANE SECTION

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A. Sommese has asked the following questions.

Questions. Let L be a very ample line bundle on a complex nonsingular surface X.

(a) When is $K_X \otimes L$ generated by its global sections?

(b) When is $K_X \otimes L$ very ample?

By using some ideas developed by Bombieri about n-connected divisors, Van de Ven ([8]) gave a very elegant proof to the following theorem.

THEOREM 1. (a) (Van de Ven, Sommese). *Let (X, L) be as above. Then $K_X \otimes L$ is generated by its global sections if and only if L is not one of the following*:

(i) *$\mathcal{O}(1)$ and $\mathcal{O}(2)$ on $\mathbf{P}^2$.*

(ii) *a line bundle on a ruled surface, the restriction of which to a fibre has degree one.*

(b) (Van de Ven). *If $h^0(X, L) \geqslant 7$ and $c_1(L)^2 \geqslant 10$, then $L \otimes K_X$ is very ample except in the following cases*:

(i) *X is a ruled surface and the restriction of L to a fibre has degree one or two.*

(ii) *X contains a nonsingular rational curve C such that $L|_C \simeq \mathcal{O}_{\mathbf{P}^1}(1)$ and $C^2 = -1$.*

In [7], Sommese gave another proof to (a) and he has studied the map associated with $|K_X \otimes L|$ in great details. Furthermore, using the Riemann Roch formula of $\mathbf{P}^4$ and the inequalities he obtained, he is also able to classify those surfaces in $\mathbf{P}^4$ with a nonsingular hyperelliptic hyperplane section.

In this paper, by combining and refining the methods of Van de Ven, Sommese and Ramanujam, we are also able to answer (b) for $h^0(L) \leqslant 6$. The main results of this paper are the following two theorems.

THEOREM 2. *Let X be a nonsingular complex projective surface and L be a very ample line bundle on X with $h^0(L) \leqslant 6$ and $c_1(L)^2 \geqslant 10$. Then $L \otimes K_X$ is very ample except in the following cases*:

(1) *X is a ruled surface and the restriction of L to a fibre has degree one or two.*

(2) *The surface contains a smooth rational curve C with $C^2 = -1$ and $L|_C \cong \mathcal{O}_{\mathbf{P}^1}(1)$.*

(3) *$h^0(L) = 5$ and $10 \leqslant c_1(L)^2 \leqslant 11$ and $h^1(\mathcal{O}_X) = 1$.*

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