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L^2-COHOMOLOGY OF LOCALLY SYMMETRIC MANIFOLDS OF FINITE VOLUME

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The L^2-cohomology space $H^{\cdot}_{(2)}(M; \mathbf{C})$ of a Riemannian manifold M may be defined as the cohomology of the complex $A^{\infty}_{(2)}(M)$ of complex valued smooth differential forms ω such that ω and $d\omega$ are square integrable (with respect to the scalar product associated to the given metric). In this paper, we consider the case where $M = \Gamma \backslash X$ is the quotient of the symmetric space X of maximal compact subgroups of a real linear semi-simple Lie group G by a discrete subgroup of finite covolume. If M is compact, then $H^*_{(2)}(M; \mathbf{C}) = H^*(M; \mathbf{C})$, so we shall always assume M to be noncompact. In that case, the relationship with the ordinary cohomology is more tenuous, but L^2-cohomology has become of interest in other contexts as well, in particular via its conjectured or proven relationships with intersection cohomology [6; 7; 8; 9; 15]. In fact, in view of such applications, we shall more generally consider the L^2-cohomology space $H^{\cdot}_{(2)}(M; \tilde{E})$ with respect to a local system $\tilde{E}$ on $\Gamma \backslash X$ associated to a finite dimensional representation (r, E) of G (endowed with an admissible scalar product).

Let $\mathscr{H}^{\cdot}_{(2)}(M; \tilde{E})$ be the space of harmonic square integrable $\tilde{E}$-valued harmonic forms. There is a natural map $j : \mathscr{H}^{\cdot}_{(2)}(M; \tilde{E}) \to H^{\cdot}_{(2)}(M; \tilde{E})$. In our case, it is injective since M is complete; moreover $\mathscr{H}^{\cdot}_{(2)}(M; \tilde{E})$ is finite dimensional [1, 4]. Then it is easily seen that $H^{\cdot}_{(2)}(M; \tilde{E})$ is finite dimensional if and only if j is bijective. Our main result is a sufficient condition for finite dimensionality, namely

THEOREM A. *The space $H^{\cdot}_{(2)}(\Gamma \backslash X; \tilde{E})$ is finite dimensional if no proper cuspidal parabolic subgroup of G contains a Cartan subgroup of a maximal compact subgroup K of G; in particular if* $\operatorname{rank} G = \operatorname{rank} K$.

(See 4.5.) Recall that a parabolic subgroup P of G is cuspidal if $\Gamma \cap R_u P$ is cocompact in $R_u P$, where $R_u P$ is the unipotent radical of P. (If G is algebraic, defined over $\mathbf{Q}$ and Γ is arithmetic, then P is cuspidal if and only if it is also defined over $\mathbf{Q}$.) It may well be that there is a converse if we allow passing to a subgroup of finite index, and if moreover E, assumed to be irreducible, is equivalent to its complex conjugate contragredient representation. In fact, if the latter condition is fulfilled and G has a minimal cuspidal subgroup which is fundamental (see 1.7), then Γ always has a subgroup of finite index such that $H^{\cdot}_{(2)}(\Gamma \backslash X; \tilde{E})$ is infinite dimensional and a (n expected) generalization of a result

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