Vol. 50, No. 3 DUKE MATHEMATICAL JOURNAL © September 1983

REGULARIZATION THEOREMS IN LIE ALGEBRA COHOMOLOGY. APPLICATIONS

A. BOREL

In computing the cohomology of a complex, it is sometimes useful to be able to replace the given complex by a smaller one without altering the cohomology. This paper proves some theorems of this type in the framework of Lie algebra or relative Lie algebra cohomology with coefficients in infinite dimensional modules. The subcomplex will usually consist of smooth vectors in the representation theoretic sense, and the passage from one to the other will involve smoothing or regularization operators, whence our title. For motivation, we first describe three applications, the first two of which are at the origin of this paper. We let G be a connected semi-simple Lie group, K a maximal compact subgroup of G, $\mathfrak{g}, \mathfrak{k}$ their Lie algebras, $X = G/K$ and Γ a torsion-free discrete subgroup of finite covolume, not cocompact since otherwise the problems considered here do not arise.

A. The cohomology $H^{\cdot}(\Gamma; \mathbf{C})$ of Γ with complex coefficients can be expressed as the cohomology of the complex $A^{\infty}(\Gamma\backslash X; \mathbf{C})$ of smooth complex valued differential forms on $\Gamma\backslash X$. Lifting those forms on $\Gamma\backslash G$ yields an isomorphism of $A^{\infty}(\Gamma\backslash X; \mathbf{C})$ with the relative Lie algebra complex $C^{\cdot}(\mathfrak{g}; \mathfrak{k}, C^{\infty}(\Gamma\backslash G))$ with coefficients in the space of complex valued smooth functions on $\Gamma\backslash G$. One wishes to replace $C^{\infty}(\Gamma\backslash G)$ by a smaller space. In [2] it is shown that we can use the space $C^{\infty}_{mg}(\Gamma\backslash G)$ of smooth functions which, together with their derivatives with respect to left invariant differential operators, have moderate growth. Here we shall see that we can reduce it further to the space $C^{\infty}_{umg}(\Gamma\backslash G)$ of functions of uniform moderate growth (i.e., the exponent limiting the growth on a Siegel set can be chosen independently of the derivatives). An application of this to the decomposition of $H^{\cdot}(\Gamma; \mathbf{C})$ was pointed out by R. P. Langlands several years ago (3.4). Actually, one would like to be able to replace $C^{\infty}_{umg}(\Gamma\backslash G)$ by automorphic forms [3; 7], but this seems to be a much harder step.

B. The L^2-cohomology of $\Gamma\backslash X$ is the cohomology of the complex $A^{\infty}_{(2)}(\Gamma\backslash X)$ of smooth forms η forms on $\Gamma\backslash X$ such that η and $d\eta$ are square integrable. It can be identified to a subcomplex $C^{\cdot}_{(2)}$ of $C^{\cdot}(\mathfrak{g}, \mathfrak{k}; C^{\infty}(\Gamma\backslash G))$ which contains $C^{\cdot}(\mathfrak{g}, \mathfrak{k}; L^2(\Gamma\backslash G)^{\infty})$. We want to prove that $H^{\cdot}(\mathfrak{g}, \mathfrak{k}; L^2(\Gamma\backslash G)^{\infty}) \to H^{\cdot}(C^{\cdot}_{(2)})$ is an isomorphism. This then gives some means to compute the latter. For applications, see [4].

C. The complexes $C^{\cdot}(\mathfrak{g}, \mathfrak{k}; L^2(\Gamma\backslash G)^{\infty})$ and $C^{\cdot}_{(2)}$ are embedded in the graded Hilbert space $C^{\cdot}(\mathfrak{g}, \mathfrak{k}; L^2(\Gamma\backslash G)) = \mathrm{Hom}_{\mathfrak{k}}(\Lambda(\mathfrak{g}/\mathfrak{k}), L^2(\Gamma\backslash G))$. The closures of d operating on $C^{\cdot}_{(2)}$ and on $C^{\cdot}(\mathfrak{g}, \mathfrak{k}; L^2(\Gamma\backslash G)^{\infty})$ are the same. We shall prove that

Received November 22, 1982.