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SUBHARMONICITY OF THE LYAPONOV INDEX

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1. Introduction. There has been intense current interest in a class of one dimensional Schrödinger operators

$$\frac{-d^2}{dx^2} + V_\omega(x) \tag{1.1}$$

on $L^2(-\infty, \infty)$ and their discrete analogs on $l^2(Z)$

$$(Mu)(n) = u(n+1) + u(n-1) + V_\omega(n)u(n) \tag{1.2}$$

where the potential V is an ergodic process in the sense that the index ω lies in a probability measure space $(\Omega, d\mu_0)$ which supports a group τ_x ($x \in R$ in case (1.1) or $x \in Z$ in case (1.2)) of measure preserving ergodic transformations with $V_\omega(x+y) = V_{\tau_y\omega}(x)$, where $\sup\{|V_\omega(x)| \,|\, x \in R \text{ or } Z,\ \omega \in \Omega\} < \infty$. The most heavily studied cases are the "random" ones where τ_x has strong mixing properties (e.g., i.i.d.'s in case (1.2) [8, 3] or Morse functions composed with Brownian motion on a compact manifold in case (1.3) [4, 9, 2]) and the almost periodic case where Ω is a compact metric space and the τ's are isometric (see [12] for a review of this).

The present paper represents a contribution to this theory. Motivated in part by old work of Thouless [13], and in part by recent work of Hermann [5] (see below), we will prove that a basic quantity is a subharmonic function, and more significantly, derive some important consequence of this observation. Interestingly enough, the fact that certain functions are upper semicontinuous while others are not will play a major role. For this reason, we single out functions which are subharmonic except for semicontinuity:

Definition. A function, f, on C with values in $[-\infty, \infty)$ is called *submean* if and only if for all $z_0 \in C$ and $r > 0$ we have that

$$f(z_0) \leqslant (2\pi)^{-1} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta. \tag{1.3}$$

For the reader's convenience we recall

Definition. A function f on C is called *uppersemicontinuous* (u.s.c.) if and only if for any $z_n \to z_\infty$, $\overline{\lim}_{n\to\infty} f(z_n) \leqslant f(z_\infty)$. Equivalently, if given z_∞ and ϵ we can find δ with $f(z) < f(z_\infty) + \epsilon$ if $|z - z_\infty| < \delta$.

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