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ALMOST PERIODIC SCHRÖDINGER OPERATORS II. THE INTEGRATED DENSITY OF STATES

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1. Introduction. In this paper, we will study Schrödinger operators, $-\Delta + V$, on $L^2(R^\nu)$ where V is an almost periodic function on R^ν. We will be especially interested in the case $\nu = 1$ where we will also consider the finite difference analog on $l^2(\mathbf{Z})$

$$(Mu)_n = u_{n+1} + u_{n-1} + V(n)u_n \tag{1}$$

where V is an almost periodic function on $\mathbf{Z}$ (Jacobi matrix). Due to the recent discoveries of several workers including G. André, S. Aubry, M. Az'bel, J. Bellissard, V. Chulaevsky, E. Dinaburg, A. Gordon, D. Hofstader, R. Johnson, J. Moser, P. Sarnak, Ya. Sinai and D. Testard as well as the present authors (see the review [20]), it has become increasingly clear that these operators have subtle and fascinating spectral properties. In this paper, we study two technical objects —the integrated density of states (ids), $k(E)$, and the Lyaponov index, $\gamma(E)$, and their relation. The most interesting consequence of these developments is found in Section 7: Explicit, simple examples of Jacobi matrices, M, with purely singular continuous spectral measures.

Let us now describe k in the Jacobi matrix case. Let $\mathscr{X}_l$ be the operator of multiplication by the characteristic function of $\{n \mid -l \leqslant n \leqslant l\}$. Then, we define

$$k(E) = \lim_{l \to \infty} (2l+1)^{-1} \mathrm{Tr}(\mathscr{X}_l P_{(-\infty, E)}(M)) \tag{2}$$

where $\dot{P}_\Omega(\cdot)$ is the spectral projection for the operator $\cdot$ on the interval Ω. We will prove the existence of the limit (2) in Section 2 in two steps: First, define the measure $d\mu_l$ by:

$$\int f(E)\, d\mu_l = (2l+1)^{-1} \mathrm{Tr}(\mathscr{X}_l f(H))$$

for *continuous* functions, f. We will show that $d\mu_l$ has a weak limit, dk, as $l \to \infty$. Secondly, we will show that dk has no pure point piece from which (2) follows with

$$k(E) = \int_{-\infty}^{E} dk$$

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