Vol. 50, No. 1 DUKE MATHEMATICAL JOURNAL © March 1983

RIEMANN DOMAINS AND ENVELOPES OF HOLOMORPHY

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1. Introduction. The envelope of holomorphy of a domain (connected open set) D in $\mathbf{C}^N$ is a Stein Riemann domain; i.e., a pair $(E(D), \eta_D)$ consisting of a Stein manifold $E(D)$ and a locally biholomorphic map $\eta_D : E(D) \to \mathbf{C}^N$. A well-known example due to Thullen [8] shows that the projection $\eta_D(E(D))$ need not be a domain of holomorphy in $\mathbf{C}^N$. This suggests a very natural question, which was part of the motivation for the present work: Which domains in $\mathbf{C}^N$ are the projection of the envelope of holomorphy of some subdomain?

This question has been considered by Fornæss and Stout [2], who showed that every domain in $\mathbf{C}^N$ is the projection of some Stein Riemann domain (M, μ). In fact, M may be chosen to be a polydisk, and μ may be chosen to have finite fibers, with the number of points in a fiber bounded by $(2N + 1)4^N + 2$. (This bound on the number of points in a fiber has been improved by Bushnell [1]. Fornæss and Stout [3] have also subsequently shown that M may be chosen to be a ball.) Fornæss and Stout were unable to decide, however, whether (M, μ) could be chosen to be the envelope of holomorphy of a domain in $\mathbf{C}^N$, and raise the question: Is every Stein Riemann domain the envelope of holomorphy of a domain in $\mathbf{C}^N$?

The other motivation for the present work comes from a desire to understand the topological relationships betwen D and $E(D)$ or $\eta_D(E(D))$. Kerner has shown [6] that the natural mapping $\pi_1(D) \to \pi_1(E(D))$ between fundamental groups is a surjection. In the same vein, Royden [7] showed that the natural maps $H^1(\overline{E(D)}, Z) \to H^1(\overline{D}, Z)$ between first Cech cohomology groups is an injection. Either of these results may be interpreted as saying that, in passing from a domain to its envelope of holomorphy, we grow no "one-dimensional holes." It is natural to expect that analogous results should hold for higher homotopy and cohomology groups. It is also natural to hope that there should be some topological relationship between the domain D and the projection $\eta_D(E(D))$.

In Section 2 of this paper we show that there are many Stein Riemann domains (M, μ) which are not the envelope of holomorphy of a domain in $\mathbf{C}^N$, thus answering in the negative the above question of Fornæss and Stout. It is

Received July 7, 1982. Revision received November 9, 1982. John-Erik Fornæss, supported in part by grants from the National Science Foundation and the Alfred P. Sloan Foundation. William R. Zame, supported in part by grants from the National Science Foundation.