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GREAT CIRCLE FIBRATIONS OF THE THREE-SPHERE

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The problem is to understand the different ways in which the three-sphere can be fibered by great circles.

In the classical example, S^3 is fibered by unit circles in the complex lines of the standard complex structure on R^4. Roughly fifty years ago, Heinz Hopf showed that this fibration provided a map from S^3 to S^2 which could not be homotoped to a constant. Allowing orthogonal changes of coordinates, one obtains other great circle fibrations of S^3, each known as a Hopf fibration. More general linear changes of coordinates yield the "skew" Hopf fibrations. All of these turn out to be just a small sample of the possible great circle fibrations of S^3.

In this paper, we catalogue such fibrations by introducing as a moduli space the space of distance decreasing maps of a two-sphere to itself. This catalogue helps us to exhibit a deformation retraction of the space of all great circle fibrations of the three-sphere to the subspace of Hopf fibrations. It also leads to a Borsuk–Ulam type result: any great circle fibration of the three-sphere must contain some orthogonal pair of circles.

More generally, we are interested in fiberings of the n-sphere by great k-spheres. Our interest is propelled by the Blaschke Problem in Differential Geometry, which seeks to characterize the simplest spaces (spheres and projective spaces) by the gross behavior of their geodesics as they emanate from a random point and coalesce again on the far side of the space. A Blaschke manifold is one whose geodesics mimic (in a certain precise sense) those on a sphere or projective space. In the case of spheres, one knows by the work of Green, Berger, Kazdan, Weinstein and Yang [Gr, B, K_1, K_2, W, Y_1, Y_2] that Blaschke spheres are isometric to round spheres.

If one knew that any fibering of the n-sphere by great k-spheres were equivalent to the Hopf fibration, one could conclude that the corresponding Blaschke manifold is at least homeomorphic to its model space. In [G–W–Y] we do exactly this for Blaschke manifolds of dimension $\leqslant 9$, as well as for those "resembling" the Cayley projective plane (a 16-dimensional space). The arguments used there exploit the

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