Vol. 49, No. 3 DUKE MATHEMATICAL JOURNAL © September 1982

GAP THEOREMS FOR NONCOMPACT RIEMANNIAN MANIFOLDS

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§0. Introduction. The curvature behavior of complete Riemannian metrics on R^2 is quite arbitrary. In particular, there are complete metrics of everywhere nonnegative curvature which have curvature zero everywhere except on a nonempty compact set: a smoothly capped half-cylinder is an example. Similarly, there are metrics of nonpositive curvature which have curvature zero except on a nonempty compact set (cf. [GW 4], Proposition 4.2 for detailed construction and further generalities). The purpose of this paper is to prove a group of theorems which together show that the complete metrics on R^n, $n \geqslant 3$, have much more restricted curvature properties than those on R^2. The first theorem states, in particular, that the nonnegative curvature phenomenon just noted in the dimension two case does not occur in higher dimensions:

THEOREM 1. *If M is a complete noncompact Riemannian manifold which is simply connected at infinity and has nonnegative sectional curvature, and if M has sectional curvature zero outside some compact set, then M is isometric to R^n.*

Here a noncompact manifold M is said to be simply connected at infinity if for any compact set $K \subset M$ there is a compact set $\hat{K}$ with $K \subset \hat{K} \subset M$ and with $M - \hat{K}$ (connected and) simply connected.

It will be shown in many cases that a complete metric of sectional curvature of constant sign (or zero) cannot even have curvature going rapidly to zero, unless its curvature is identically zero, i.e. unless the metric is flat. To state these results precisely, the following conventions and terminology will be used: M is to be complete noncompact Riemannian manifold of dimension n, $0 \in M$ a chosen point of M and ρ the Riemannian distance function on M, $\rho(q) = \mathrm{dis}_M(0, q)$, $q \in M$. Define $k: [0, +\infty) \to \mathsf{R}$ by $k(s) = \sup\{|\text{sectional curvature at } q| : q \in M, \rho(q) = s\}$. Philosophically, there perhaps seems at first sight to be no reason to separate even from odd dimensions except for the specialness of dimension two already noted. However, technical differences exist, especially in the proofs, between the even and odd-dimensional cases, arising essentially from the fact that the generalized Gauss–Bonnet theorem ([AW]) is meaningful only in even dimensions. It is thus convenient to state the detailed results separately for even and odd dimensions and to state some specialized cases as well (e.g., Theorem 5). General explanations of these technical differences will be given after the

Received December 23, 1981. Research of both authors supported in part by the National Science Foundation.