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ON THE NUMBER OF CLOSED GEODESICS ON A COMPACT RIEMANNIAN MANIFOLD

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A well-known theorem of Gromoll and Meyer [4] states that any Riemannian metric on a compact, simply connected manifold M has infinitely many geometrically different closed geodesics if the Betti numbers of the free loop space of M are not bounded. In [5] Gromov proved a quantitative version of this result under the additional assumption that all closed geodesics of the metric are non-degenerate. This is a generic condition on the metric by the bumpy metric theorem of Abraham [1]. In this note we will improve Gromov's estimate.

Let Λ be the space of piecewise differentiable closed curves $c : \mathsf{R}/\mathsf{Z} \to M$, endowed with the compact-open topology. For a principal ideal domain R denote by $b_k(R)$ the rank of $H_k(\Lambda; R)$.

For a Riemannian metric g on M and $t \geqslant 0$ define $N_g(t)$ to be the number of geometrically different closed geodesics of g of length $\leqslant t$. Gromov proved that there exist constants $\alpha = \alpha(g) > 0$ and $\beta = \beta(g) > 0$ such that $N_g(t) \geqslant \alpha(\sum_{k<\beta t} b_k(R))/t$ for all t sufficiently large. We will prove:

THEOREM. *Suppose M is compact and simply connected. Let g be a Riemannian metric on M such that all closed geodesics of g are nondegenerate. Then there exist constants $\alpha = \alpha(g) > 0$ and $\beta = \beta(g) > 0$ such that*

$$N_g(t) \geqslant \alpha \max_{k \leqslant \beta t} b_k(R)$$

for any principal ideal domain R and all t sufficiently large.

Proof of the Theorem. Let g be a Riemannian metric on M. The closed geodesics of g and the point curves are the critical points of the energy functional

$$E : \Lambda \to \mathsf{R}, \qquad E(c) = \frac{1}{2}\int_0^1 g(\dot{c}, \dot{c}).$$

Let $L : \Lambda \to R$ denote the length functional. Applying the Cauchy–Schwarz inequality we get $L^2(c) \leqslant 2E(c)$, where equality holds if and only if c is parametrized proportional to arc-length.

Set $S = \mathsf{R}/\mathsf{Z} = [0,1]/\{0,1\}$. We have an S action on Λ: if $s \in S$ and $c \in \Lambda$, then sc is defined by $sc(t) = c(t+s)$. The orbit of c is denoted by Sc. If c is not a

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