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PROPER HOLOMORPHIC MAPPINGS FROM STRONGLY PSEUDOCONVEX DOMAINS

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1. Introduction. Let Ω, $D \subset\subset \mathbf{C}^n$ be open sets. A holomorphic mapping is *proper* if $f^{-1}(K)$ is compact whenever $K \subset D$ is compact. Our result is that when Ω is strongly pseudoconvex and simply connected, a proper mapping f gives rise to a biholomorphic map as follows.

THEOREM. *Let $\Omega \subset\subset \mathbf{C}^n$, $n \geqslant 2$, be a simply connected, strongly pseudoconvex domain with C^2 boundary. If $f: \Omega \to D$ is a proper mapping, and if $f \in C^\infty(\overline{\Omega})$, then there is a finite subgroup $\Gamma \subset Aut(\Omega)$ with the properties:*

(i) *$f(g(z)) = f(z)$ for all $g \in \Gamma$;*

(ii) *for $z_1, z_2 \in \Omega$ with $f(z_1) = f(z_2)$, there exists $g \in \Gamma$ with $g(z_1) = z_2$.*

(iii) *if $\eta: \Omega \to \tilde{\Omega} = \Omega/\Gamma$ is the quotient map, and $\tilde{f}: \tilde{\Omega} \to D$ is the mapping induced by f, then $\tilde{f}$ is a biholomorphism, and $\tilde{f}\eta = f$.*

Remark 1. The motivation behind the Theorem is that it is "difficult" for a proper mapping not to be biholomorphic. Note that biholomorphic mappings always exist in abundance, e.g., we may let f be a small C^1 perturbation of the identity mapping $i(z) = z$ and $D = f(\Omega)$. On the other hand, if f is smooth but not one-to-one, then the automorphism group of Ω must be nontrivial. But it is known (see Burns, Shnider, and Wells [1]) that a "generic" strongly pseudoconvex domain has no automorphisms except the identity. Thus if Ω is such a domain and $\pi_1(\Omega) = 0$, every proper mapping $f: \Omega \to f(\Omega)$, $f \in C^\infty(\overline{\Omega}) \cap \mathcal{O}(\Omega)$ is biholomorphic.

Remark 2. A number of results in this direction are known already. If $\partial\Omega$ is real analytic, then the existence of the group Γ follows from a result of Pinčuk [3].

Pinčuk [2] also obtains an interesting result in the case where ∂D is smooth. The Theorem above, however, is uninteresting in this case. For if f is not locally biholomorphic, then by the Corollary below, ∂D is not smooth.

In the case where $\Omega = \mathbf{B}^n$ is the unit ball in $\mathbf{C}^n$, then the Theorem above was obtained by Rudin [6], who also gives a more detailed study of the possible quotient maps $\eta: \mathbf{B}^n \to \mathbf{B}^n/\Gamma$ that can arise from a proper mapping.

Remark 3. Without changing the proof, we may assume that $\Omega \subset\subset \hat{\Omega}$ is a smoothly bounded, strongly pseudoconvex domain in a Stein manifold $\hat{\Omega}$ and that $D \subset\subset \hat{D}$, where $\hat{D}$ is any complex manifold.

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