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KUMMER AND HERBRAND CRITERION IN THE THEORY OF FUNCTION FIELDS

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Introduction. One of the most beautiful and profound aspects of the theory of cyclotomic fields is Kummer's criterion. This says that for p odd, p divides the class number of $\mathbb{Q}(\zeta_p)$ if and only if p divides the numerator of some Bernoulli number B_{2K}, for $K = 1, \ldots, p - 3/2$. It is the purpose of this paper to give the appropriate version of Kummer's criterion in the arithmetic theory of function fields. Like its cyclotomic analog, this result also determines when a certain class number is divisible by p, p being the characteristic of the function field.

For the rational field, $k = \mathbb{F}_r(T)$, $r = p^n$, the result is the following: Let $f \subseteq F_r[T]$ be a prime of degree d. In §1 we define an abelian extension, $k(f)$, of k which is modeled on $\mathbb{Q}(\zeta_p)$ and with class number h_f. In §3 we define a certain function $\beta : \mathbb{N} \to \mathbb{F}_r[T] - \{0\}$ which is closely related to the zeta-function of $\mathbb{F}_r[T]$ (defined in [5], [7]). We then show

$$p \mid h_f \qquad \text{if and only if} \quad f \left| \prod_{i=1}^{r^d-2} \right| \beta(i).$$

Actually, as is classical, the class number h_f splits up into h^+h^-. The minus part is handled by the product over $i \equiv 0(r-1)$ and the plus part is handled by the rest. As of now, there is no known relationship between the two.

As a corollary of our results, we can show the analog of classical results such as the existence of infinitely many primes with p dividing h^-.

In the theory of cyclotomic fields, Herbrand's theorem represents a partial strengthening of Kummer's criterion; it says that certain isotypic parts of the p-part of the class group are trivial. Using recent work of Gross and Tate, we are also able to give an appropriate version of this in the rational field case. Combining this with some work of E. Thomas, we also can show certain isotypic parts vanish.

The Herbrand criterion raises some extremely fascinating possibilities: Our definition of $k(f)$ comes from the theory of elliptic modules (see [1], [2], [4]). The theory at the positive integers of the author's functions is also directly related to this theory (see [6]). On the other hand, the theory at the negative integers is not, a-priori, related to this theory; yet it is the one that comes up here. Therefore, if there is to be a version of Ribet's converse to Herbrand, which uses GL(2), then the author's functions must satisfy some unknown functional equation. Some

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