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EXPONENTIAL LOCALIZATION IN ONE DIMENSIONAL DISORDERED SYSTEMS

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1. Introduction. The study of spectral properties of random operators is an old and quite interesting mathematical problem which deserves to be investigated for its own sake. Our concern in this field comes from the new wave of interest initiated by recent works of the Russian school (see for example [7] and [14]) which put on firm and rigorous mathematical grounds the famous works of the physicists Anderson and Mott on the localization of states in one dimensional disordered quantum systems. The result is the following: let $\{q(t);\ t \in \mathbb{R}\}$ be a stochastic process which satisfies some restrictive conditions to be specified later on (see section 3.2 below), then almost surely, the random self-adjoint operator:

$$\mathbf{H} = -\frac{d^2}{dt^2} + q(t) \tag{1.1}$$

on $L^2(\mathbb{R}, dt)$ has pure point spectrum (i.e., no continuous component) and the eigenfunctions fall-off exponentially as $|t|$ goes to infinity.

Such an unexpected spectral behavior is typical of the randomness of the operator and there is an evident need for a deep understanding of the drastic changes due to this randomness. Unfortunately the proofs of [7] and [14] are lengthy and technical. One of our objectives is to provide the newcomers to this subject with a simpler proof of this important result. Actually, a simple estimate (lemma 3.3 below) appears to be the cornerstone of our argument. It immediately implies that almost surely:

(i) the spectrum is pure point,

(ii) the eigenfunctions fall off exponentially,

(iii) the rate of exponential fall off is given by the upper Lyapunov exponent of the corresponding random Cauchy problem.

Results (i) and (ii) were proved in [7] and [14] respectively, but result (iii) seems to appear for the first time. It is very natural and was generally expected to be true. It was formulated as a conjecture by S. A. Molčanov in [14] who justified it by a fine study of the localization of the eigenfunctions in bounded intervals.

The weakness of our method is that it is limited to the one dimensional case

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