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TWO BOUNDARIES OF TEICHMÜLLER SPACE

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§1. Notation and statement of the theorem. In this paper we prove that the Teichmüller and Thurston compactifications of Teichmüller space are almost everywhere, but not everywhere the same. That they are different was first proved by Kerckhoff. The precise statement of the theorem after the appropriate notation appears below. We rely heavily on the Thurston theory of measured foliations and Teichmüller space, and the theory of quadratic differentials and Teichmüller extremal maps. The reader is referred to [2], [3], [4], and [5]. We will briefly recall the ideas, leaving many of the details for the references.

Suppose X is a closed Riemann surface of genus $g \geqslant 2$, $H^0(X, \Omega^{\otimes 2})$ the Banach space of holomorphic quadratic differentials on X with $\|\varphi\| = \int_X |\varphi|$, B_1 the open unit ball and Q_0 the unit sphere in $H^0(X, \Omega^{\otimes 2})$. For each $\varphi \in B_1$ consider the pair (X_φ, f_φ) where $f_\varphi : X \to X_\varphi$ is the Teichmüller extremal map with dilation $\|\varphi\| \bar{\varphi}/|\varphi|$. The map of B_1 to the Teichmüller space T_g given by

$$\varphi \to (X_\varphi, f_\varphi)$$

is a homeomorphism. This realization of T_g by B_1 is called the Teichmüller embedding; Q_0 is the natural boundary. We denote this Teichmüller compactification by $\overline{T}_g$.

Next consider T_g as the space of metrics of curvature -1 up to isometries isotopic to the identity on a fixed C^∞ surface M. Let S be the set of homotopy classes of simple closed curves on M with the discrete topology; R_+^S is given the product topology and PR_+^S is the corresponding projective space. The map $T_g \mapsto PR_+^S$ defined by $[\mu] \to (\gamma \to [\mu](\gamma))$ where $[\mu]$ is an equivalence class of metrics and $[\mu](\gamma)$ is the length of the unique geodesic in the class of γ is injective. The map to PR_+^S is still injective and is called the Thurston embedding of Teichmüller space. The boundary in PR_+^S is the sphere PF of projective measured foliations on M. The union of T_g and PF is denoted T_g^T and is called the Thurston compactification. The space of measured foliations is denoted MF. There is an obvious homeomorphism h between the Teichmüller and Thurston realizations. Namely, for each $\varphi \in B_1$, consider X_φ as a new complex structure on X and f_φ as homotopic to the identity. Then $h(\varphi)$ is the point in PR_+^S determined by the Poincaré metric on X_φ.

Any $\varphi \in H^0(X, \Omega^{\otimes 2})$ defines a pair F_φ and $F_{-\varphi}$ of measured foliations called the horizontal and vertical foliations. They are defined locally by the 1-forms

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