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ON THE VOLUME OF A NONCOMPACT MANIFOLD

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The main purpose of this paper is to apply the method developed in Wu [3] to prove:

THEOREM. *Let M be a noncompact complete Riemannian manifold which satisfies*

$$\text{Ricci curvature} \geqslant \frac{-A}{\rho^{2+\epsilon}}$$

where ρ denotes the distance from a fixed point of M and A and ϵ are positive constants. Then M has infinite volume.

A more general theorem, omitted here because of its technical complexity, will be given in the course of the proof. One corollary of the theorem is that a complete noncompact Riemannian manifold whose Ricci curvature is nonnegative outside a compact set must have infinite volume. This corollary generalizes previously known results in this direction (Satz 4 in [Cohn-Vossen], [Huber], Wu [1] and [2], Greene–Wu [1], Yau [1], [Calabi], and Yau [2].) Note that the exponent $(2 + \epsilon)$ in the Theorem cannot be lowered to 2; a surface in $\mathbf{R}^3$ which, outside a compact set, equals the surface of revolution obtained by rotating the curve $y = 1/x^2$ $(1 \leqslant x \leqslant \infty)$ around the x-axis, has finite area and its curvature grows like $-\rho^{-2}$.

The basic idea of the proof of the Theorem, as in Greene–Wu [1], is to exploit the existence of a globally Lipschitzian subharmonic function. The most natural function that comes to mind in this connection is the Busemann function η associated with a ray. But whereas η is subharmonic when the Ricci curvature is nonnegative, it is decidedly not subharmonic in the present context since the Ricci curvature is allowed to be negative. The key observation here is that since the Ricci curvature has a good lower bound, a certain composite function $\chi(\eta)$ will be subharmonic and Lipschitzian if χ is a carefully chosen C^∞ increasing convex function of one variable. If η is C^2, then $\Delta\chi(\eta) = \chi''(\eta) \cdot |d\eta|^2 + \chi'(\eta)\Delta\eta$, so that a precise knowledge of $\Delta\eta$ would allow us to conclude that $\Delta\chi(\eta) \geqslant 0$. However, since η is in general only continuous, a new method is needed that would provide a good hold on $\Delta\eta$ as well as justify the preceding computation of $\Delta\chi(\eta)$ assuming only the continuity of η. The method developed in Wu [3] exactly fulfills this need. We will quickly review the basic facts of this method.

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