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SOME EXAMPLES OF HOMOGENEOUS EINSTEIN MANIFOLDS IN DIMENSION SEVEN

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Let $M_{k,l}$ denote the homogeneous space $SU(3)/i_{k,l}(S^1)$, where $i_{k,l}$ is the embedding of the circle in $SU(3)$ via $e^{2\pi i\theta} \mapsto \operatorname{diag}(e^{2\pi ik\theta}, e^{2\pi il\theta}, e^{-2\pi i(k+l)\theta})$. Aloff and Lashof independently noted that $H^4(M_{k,l}; \mathbf{Z}) \approx \mathbf{Z}/|k^2 + l^2 + kl|\mathbf{Z}$ (see [1]), so that the $M_{k,l}$'s have among them infinitely many homotopy types. In this paper we prove the following

PROPOSITION 1. *Let $(k, l) = 1$ and $k \not\equiv l \pmod 3$ so that $SU(3)$ acts effectively on $M_{k,l}$. Then each $M_{k,l}$ admits a homogeneous Einstein metric. The Aloff–Wallach metrics, however, are not Einstein.*

These examples have the following interesting features:

1. The isotropy representations split into three irreducible representations of the circle plus a one-dimensional trivial representation.
2. They are not naturally reductive with respect to the decomposition of $\mathfrak{su}(3)$ given in section one.
3. They are the first examples of infinitely many homotopically distinct compact homogeneous Einstein manifolds *in a fixed dimension*.
4. These metrics cannot be obtained using the Riemannian submersion $S^1 \to M_{k,l} \to SU(3)/T$, where T is the standard maximal torus of $SU(3)$, and $SU(3)/T$ is equipped with a homogeneous Einstein metric. Compare [4].

In section one, all preliminaries and notations are stated. Proposition 1 is proved in section two. In section three we show that there exists a sequence of the $M_{k,l}$'s, each with a homogeneous Einstein metric of volume 1, whose corresponding scalar curvatures become arbitrarily small. We also discuss briefly homogeneous Lorentz Einstein metrics on the $M_{k,l}$'s.

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1. Preliminaries. In the notation of the introduction let $m = -k - l$. We equip $\mathfrak{g}$, the Lie algebra of $G = SU(3)$, with the bi-invariant metric $\langle X, Y\rangle_0 = -\operatorname{Re}\operatorname{tr}(XY)$. Let $\mathfrak{h}_{k,l}$ denote the Lie algebra of $i_{k,l}(S^1) = H_{k,l}$ and $\mathfrak{t}$ the Lie algebra of the standard maximal torus T of $SU(3)$.

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