Vol. 48, No. 2 DUKE MATHEMATICAL JOURNAL © June 1981

DEFORMATIONS OF UNI-RULED VARIETIES

MARC N. LEVINE

Introduction. Let V be a variety of dimension n. We say that V is a ruled variety if V is birationally isomorphic to a product $S \times \mathbf{P}^1$, for some variety S. V is said to be uni-ruled if there is a ruled variety W of dimension n, and a dominant, rational map $f: W \to V$. V is said to be separably uni-ruled if there is an f as above that is a separable map.

Let V be a smooth separably uni-ruled variety. Our main result (Theorem 6) is that small smooth deformations of V are uni-ruled. If V is defined over a field of characteristic zero, then all smooth deformations of V are uni-ruled. We actually prove a somewhat stronger result, in that we only require that V be an algebraic space. We combine this result with a classification result of Viehweg [8], and show that the set of smooth irregular threefolds in $\mathbf{P}^N_{\mathbf{C}}$, of Kodaira dimension minus infinity, forms an open and closed subset of the Hilbert scheme of smooth threefolds in $\mathbf{P}^N_{\mathbf{C}}$.

We use the following notations and conventions:

If M is a scheme, V an algebraic space, and $f: V \to M$ a morphism, separated and of finite type, with geometrically irreducible fibers, we call V an M-space. If M is the spectrum of a ring R, we will call V an R-space. If K is a field, and V a K-space that is birational to a ruled (resp. uni-ruled, resp. separably uni-ruled) variety, we call V a ruled (resp. uni-ruled, resp. separably uni-ruled) K-space.

If V and W are algebraic spaces, separated and of finite presentation over a scheme M, we let $\mathrm{Hom}_M(V, W)$ denote the algebraic space over M representing the functor $\underline{\mathrm{Hom}}_M(V, W)$,

$$\underline{\mathrm{Hom}}_M(V, W)(S) = \{ S\text{-morphisms } f: V \times_M S \to W \times_M S \}$$

for S an M-scheme. By [1], $\mathrm{Hom}_M(V, M)$ is an algebraic space over M, separated and locally of finite presentation. If $f: V \times_M S \to W \times_M S$ is an S-morphism, we let $[f]$ denote the S-valued point of $\mathrm{Hom}_M(V, W)$ canonically associated to f.

We fix an algebraically closed field k; unless mentioned otherwise, all algebraic spaces, morphisms, and rational maps are over k.

If x is a point of k-space X, we let $T_x(X)$ denote the Zariski tangent space to X at x. On X, $\Omega^1_{X/k}$ will denote the sheaf of Kahler one forms on X; if X is smooth over k, we let Θ_X denote the tangent sheaf of X. If B is an algebraic space we let $|B|$ denote the reduced algebraic space associated to B.

Received October 31, 1980.