Vol. 47, No. 4 DUKE MATHEMATICAL JOURNAL © December 1980

ELLIPTIC AND SUBELLIPTIC ESTIMATES FOR OPERATORS IN AN ENVELOPING ALGEBRA

ROE GOODMAN

Introduction. Let G be a Lie group with Lie algebra $\mathfrak{G}$, and denote by $U(\mathfrak{G})$ the complexified universal enveloping algebra of $\mathfrak{G}$, with its usual filtration $\{U_m\}_{m\geqslant 0}$ (cf. [2]). Given a strongly continuous unitary representation π of G on a Hilbert space $\mathcal{H}(\pi)$, we let $(\pi_{\pm\infty}, \mathcal{H}^{\pm\infty})$ be the associated representations of $U(\mathfrak{G})$ on the space $\mathcal{H}^\infty$ of C^∞ vectors and its dual $\mathcal{H}^{-\infty}$ [1]. In this paper we study regularity properties of operators $\pi_{-\infty}(T)$, $T \in U(\mathfrak{G})$, which satisfy elliptic or subelliptic estimates relative to a natural G-invariant interpolating scale of "Sobolev" spaces $\mathcal{H}^t$, $-\infty \leqslant t \leqslant \infty$. (When t is a positive integer, $\mathcal{H}^t$ is the space of t-times differentiable vectors for π.) Typical examples of such operators are a "Laplacian" $\Delta = \sum Y_i^2$ and a "subelliptic Laplacian" $J = \sum X_i^2$. Here $\{Y_i\}$ is a basis for $\mathfrak{G}$, while $\{X_i\}$ is only assumed to generate $\mathfrak{G}$ as a Lie algebra.

In the elliptic case we studied this problem in [9], to which we refer for citations of previous literature and applications. The main contribution of the present paper in this direction is the introduction and utilization of an algebra of "pseudo-differential" operators. This algebra is obtained by adjoining to $U(\mathfrak{G})$ the operators Λ^t, $t \in \mathbf{R}$, where $\Lambda = (I - \Delta)^{1/2}$. Unifying and simplifying our previous work, (cf. [7], [8], [9]), we give in §§1–2 a relatively self-contained exposition of estimates for these operators and their commutators. This is applied in §3 to obtain the regularity theorem for a π-*coercive* operator $T \in U_{2m}(\mathfrak{G})$ and properties of the semi-group generated by $\pi(T)$. (We also fill in an apparent gap in the proof of a key lemma in [9].)

It should perhaps be pointed out that the "obvious" attempt to prove Theorem 3.1 when T is elliptic by applying standard elliptic regularity theory to the action of T on the matrix entries of the representation fails, in the following rather subtle way: the classical, potential-theoretic elliptic estimates on the Lipschitz spaces $\mathrm{Lip}(\alpha)$ break down when the Lipschitz exponent α is an integer, as is well known. On the other hand, we proved ([9], Cor. 5.1) that the elements of $\mathcal{H}^k$, $k \in \mathbf{N}$, correspond to representative functions in the space $\mathrm{Lip}(k)$. Thus if $u \in \mathcal{H}$, $v \in \mathcal{H}^k$ and $\pi_{-\infty}(T)u = v$, the entry functions $f(x) = (\pi(x)u, w)$ and $g(x) = (\pi(x), v, w)$ satisfy the equation $Tf = g$ in the weak sense, for any $w \in \mathcal{H}$. (T acting as a left-invariant differential operator on G). One has $g \in C^k(G)$, but as noted above, this is not sufficient to imply that $f \in \mathrm{Lip}(k + 2m)$, which is needed to show that $u \in \mathcal{H}^{k+2m}$.

Received September 11, 1979. Revision received June 12, 1980. Research partially supported by NSF Grant MCS 76-07097.