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STRICTLY NON-ERGODIC ACTIONS ON HOMOGENEOUS SPACES

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In [4], J. Glimm has discussed several equivalent conditions under which the space of orbits of a locally compact transformation group can be said to be 'nicely behaved.' These conditions were also studied in a wider context by E. G. Effros and have found applications to C^*-algebras and their representations [3]. In a certain sense these conditions are opposite to strict ergodicity and hence a transformation group satisfying them may be said to be *strictly non-ergodic*.

Let G be a connected Lie group and Γ be a lattice in G; that is, Γ is a discrete subgroup and G/Γ admits a finite G-invariant measure. Let H be a closed compactly generated subgroup of G. In this note we obtain a necessary and sufficient condition for the action of H on G/Γ to be strictly non-ergodic. Each orbit being locally closed is one of the equivalent conditions for strict non-ergodicity. Hence the action of a compact subgroup is strictly non-ergodic. On the other hand if V is a normal subgroup of G such that $V\Gamma$ is closed, then the action of V on G/Γ is strictly non-ergodic. We show that any closed compactly generated subgroup H acting strictly non-ergodically is built up from these two types, in the sense that H contains a subgroup V such that V is normal in G, $V\Gamma$ is closed and H/V is compact (cf. Theorem 2.1).

Let G and Γ be as above and let H be any closed subgroup of G. A point $x \in G/\Gamma$ is said to be *H-periodic* if Hx is closed and admits a finite H-invariant measure. We show that if the set of H-periodic points has positive outer measure (with respect to the G-invariant measure on G/Γ) then the action of H is strictly non-ergodic and satisfies the above condition. Using this we deduce that if H is a finitely generated discrete subgroup then 'nice behaviour' of the orbit structure on any H-invariant set of positive measure (rather than everywhere) already implies strict non-ergodicity (cf. Theorem 3.1).

The author was led to consider these questions because of [9], where a similar investigation (of nice behaviour of the orbit structure on a set of positive measure) was carried out for a rather restrictive class of subgroups; viz. the subgroup H as above was also required to be a lattice in G. Our results generalize the 'main theorem' of [9] and put it in a wider perspective.

§1. Strict non-ergodicity. The following theorem enumerates some of the conditions on the space of orbits whose equivalence is proved in [3] and [4]. For obvious reasons we do not strive for complete generality in their statement.

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