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NON-SQUARE-INTEGRABLE COHOMOLOGY OF ARITHMETIC GROUPS

AVNER ASH

§1. Introduction. Suppose G is a semi-simple algebraic group defined over $\mathbf{Q}$, K a maximal compact subgroup of $G(\mathbf{R})$, and $\Gamma \subset G(\mathbf{Q})$ a torsion-free arithmetic subgroup. Then $X = K \backslash G(\mathbf{R})$ is a riemannian symmetric space with a $G(\mathbf{R})$-invariant metric, unique up to scalar multiplication. Our group Γ acts freely and properly on X, so that X/Γ is a smooth manifold whose inheritance of the metric on X makes it a riemannian locally-symmetric manifold. The metric extends to an inner product on the covectors of a given dimension at a given point. We call a differential form α "square-integrable" if and only if

$$\int_{X/\Gamma} \langle \alpha(x), \alpha(x) \rangle \, dx < \infty.$$

The group cohomology $H^*(\Gamma, \mathbf{R})$ of Γ with trivial Γ-module $\mathbf{R}$ as coefficients is naturally isomorphic to the de Rham cohomology $H^*(X/\Gamma, \mathbf{R})$ of X/Γ, and we will identify the two. We define $H^*_{(2)}(\Gamma)$ to be the subgroup of those cohomology classes which can be represented by square-integrable differential forms on X/Γ. These square-integrable forms can be investigated using techniques of the theory of Hilbert space representations of Lie groups.

Hence it becomes of interest to discover what gap, if any, exists between $H^*_{(2)}(\Gamma)$ and $H^*(\Gamma)$. A theorem of Garland, strengthened by Borel in [1], gives an integer $c(G)$ depending only on the $\mathbf{Q}$-group G, such that $H^i_{(2)}(\Gamma) = H^i(\Gamma)$ for $i \leqslant c(G)$. This constant tends to be small compared with the cohomological dimension of Γ. For instance, if $G = SL(n)$ with the usual $\mathbf{Q}$-structure, $c(G)$ is approximately $n/2$.

The object of this paper is to exhibit some non-square-integrable cohomology classes for certain Γ's. As far as I know, no examples are known whose dimension is less than the $\mathbf{R}$-rank of G. Some of my examples, for instance when $G = SL(n)$ as above, occur at the $\mathbf{R}$-rank.

Our main theorem: Suppose P is a $\mathbf{Q}$ parabolic subgroup of G, that Γ is neat, and that Γ satisfies an additional hypothesis relative to P, spelled out in section 2. Any arithmetic subgroup of G will contain a subgroup of finite index satisfying these hypotheses. Let d be the dimension of U, the unipotent radical of P. Let Γ_0 be a subgroup of finite index in Γ and set $e(\Gamma_0)$ to be the number of

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