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ON THE φ-MIXING CONDITION FOR STATIONARY RANDOM SEQUENCES

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Summary. Let $\varphi_1, \varphi_2, \varphi_3, \ldots$ be the dependence coefficients associated with the φ-mixing condition for a given strictly stationary random sequence. If the sequence is mixing, then either $\varphi_n \to 0$ or $\varphi_n \equiv 1$. An extension is given of a theorem of Kesten and O'Brien on the rate at which φ_n can approach 0.

Let $(X_k, k = \ldots, -1, 0, 1, \ldots)$ be a strictly stationary sequence of real-valued random variables on a probability space $(\Omega, \mathfrak{F}, P)$. For any collection Y of r.v.'s let $\mathfrak{B}(Y)$ be the Borel field generated by Y. For $-\infty \leq J \leq L \leq \infty$ let $\mathfrak{F}_J^L = \mathfrak{B}(X_k, J \leq k \leq L)$. For any two σ-fields $\mathfrak{A}$ and $\mathfrak{B}$ define

$$\varphi(\mathfrak{A}, \mathfrak{B}) = \mathrm{Sup}|P(B \mid A) - P(B)| \qquad A \in \mathfrak{A}, B \in \mathfrak{B}, P(A) > 0.$$

For each $n \geq 1$ let $\varphi_n = \varphi(\mathfrak{F}_{-\infty}^0, \mathfrak{F}_n^\infty)$. The φ-mixing condition of I. A. Ibragimov (1959) is: $\varphi_n \to 0$ as $n \to \infty$.

Starting with Ibragimov [1], numerous limit theorems and invariance principles have been proved under the φ-mixing condition, sometimes requiring additional conditions on the rate at which $\varphi_n \to 0$; see, for example, the results referred to on pp. 26–29 of W. Philipp and W. Stout [4]. In the case where (X_k) is an aperiodic Markov chain with countable irreducible state-space, if $\varphi_n < 1$ for some n then $\exists C > 0$ and $a > 0$ such that $\varphi_n \leq Ce^{-an}$ $\forall n$ (see M. Rosenblatt [5], pp. 209–212). It was once an open question whether φ_n had to approach 0 exponentially fast in the general case, but then H. Kesten and G. L. O'Brien [3] showed that on the contrary φ_n can approach 0 arbitrarily slowly.

Before stating our results we need some definitions. We assume T is an $\mathfrak{F}$-measurable P-measure-preserving automorphism of Ω, and U is the operator on $\mathfrak{F}$-measurable r.v.'s defined by $Uf(w) = f(Tw)$. We assume that $X_n = U^n X_0$ for every integer n. Let $S = T^{-1}$. For any event A let I_A denote its indicator function; if $A \in \mathfrak{F}_J^L$, then $SA \in \mathfrak{F}_{J+1}^{L+1}$ and $UI_A = I_{SA}$.

Also, for each $n \geq 1$ let $\varphi_n^* = \varphi(\mathfrak{F}_n^\infty, \mathfrak{F}_{-\infty}^0)$; these φ_n^*'s are simply the φ_n's for the sequence (Y_k) defined by $Y_k = X_{-k}$, and in their article Kesten and O'Brien called attention to an example of (X_k) for which $\varphi_n \to 0$ but $\varphi_n^* \not\to 0$. Here we will prove these two theorems:

THEOREM 1. *If (X_k) is strictly stationary and mixing ($\forall A, B \in \mathfrak{F}_{-\infty}^\infty$, $P(A \cap S^n B) \to P(A)P(B)$ as $n \to \infty$), then either $\varphi_n \to 0$ as $n \to \infty$ or $\varphi_n = 1$ $\forall n$.*

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