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INVERSE SCATTERING THEORY FOR PERTURBATIONS OF RANK ONE

PREBEN ALSHOLM

1. Introduction. Let $\mathbf{H}$ be a separable Hilbert space and A a self-adjoint operator in $\mathbf{H}$, which is spectrally absolutely continuous, i.e., if E is the spectral measure for A, then the *absolutely continuous subspace* for A (the set of all $u \in \mathbf{H}$ for which $(E(\cdot)u, u)$ is absolutely continuous with respect to Lebesgue measure) coincides with $\mathbf{H}$. Let B be a rank one perturbation of A, i.e., $Bu = Au + c(u, f)f$ for $u \in \mathbf{D}(A)$, the domain of A. We take $f \in \mathbf{H}$ to be a unit vector and the number c to be real, thus B is self-adjoint. The *wave operators* $W_\pm$ for the pair B, A are defined by $W_\pm = \text{s-lim}_{t\to\pm\infty} \exp(itB)\exp(-itA)$, where s-lim means strong limit. T. Kato [4] has proved that $W_\pm$ exist and are *complete*, i.e., in addition to the general identity

$$W_\pm^* W_\pm = I, \tag{1.1}$$

we also have that $W_\pm W_\pm^*$ is the orthogonal projection onto the absolutely continuous subspace for B. Thus *the scattering operator* $S = W_+^* W_-$ is unitary and the restriction B_{ac} of B to its absolutely continuous subspace is unitarily equivalent to A, in fact $B_{\text{ac}} = W_\pm A W_\pm^*$.

The *inverse problem* of scattering theory is the problem of constructing the perturbation, i.e., $V = B - A$, from the scattering operator S and from some information about the eigenvalues of B. In this paper we shall relate V (i.e., c and f) to the spectral shift function $\xi(x)$ of Krein. For a perturbation V of trace class ξ is given by

$$\xi(x; B, A) = \xi(x) = (1/\pi) \lim_{\epsilon\downarrow 0} \arg\det\left(1 + V(A - x - i\epsilon)^{-1}\right), \tag{1.2}$$

see Krein [5], Birman and Krein [1] or A. Jensen and T. Kato [3]. We pick that branch of the arg-function for which $\arg\det(1 + V(A - z)^{-1})$ tends to zero as $\operatorname{Im} z$ tends to infinity. In a representation in which A is multiplication by the variable x, S is diagonal (since it commutes with A) and $\det S(x) = \exp(-2\pi i\xi(x))$. In our case $S(x)$ is a complex-valued function so that $S(x) = \exp(-2\pi i\xi(x))$, and we shall prove this result directly. The discontinuities of ξ are simple (i.e., ξ has limits from both the right and the left at those points) and are located at the eigenvalues for B. If conversely we are given N real numbers, $\lambda_1, \lambda_2, \ldots, \lambda_N$, and a unitary operator S, which in a diagonal representation for

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