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SOME RESULTS IN H^p THEORY FOR THE HEISENBERG GROUP

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Virtually all of the known H^p (Hardy space) theory for R^n, save those parts involving characterization of H^p through Riesz transforms, has an analogue on the Heisenberg group H^n. It is our intention in this paper to justify this statement by presenting those techniques needed to develop the analogues. What can be done about characterization through "Riesz transforms" is not known.

At this writing, a few technical difficulties prevent us from developing the full real-variable theory of H^p when $p < 1$ (see section 6). It is to be hoped that these problems will be surmounted soon. At any rate, the case $p = 1$ is the one of greatest interest.

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1. The Laplace-Beltrami and related operators. We begin with a general discussion of the Laplace-Beltrami operator Δ and some hitherto unexplored variants. The variants appear to be of real importance in the study of Δ itself, although their use is not crucial in this paper. Corollary 1.3(b), an estimate on the derivatives of harmonic functions, could not be obtained, to our knowledge, without the variants. We use this result in the H^p theory but a weaker result which could be proven without the variants would suffice. See the discussion at the end of this section.

We let $\mathsf{U}^{n+1} = \{[z_0, z] \in \mathsf{C} \times \mathsf{C}^n \mid h = \operatorname{Im} z_0 - |z|^2 > 0\}$, the Siegel upper half space of type II; frequently we use instead the coordinates (h, u), where $u = (t, z)$, $t = \operatorname{Re} z_0$.

H^n is the Lie group with underlying manifold $\mathsf{R} \times \mathsf{C}^n$ and multiplication $(t, z) \cdot (t', z') = (t + t' + 2 \operatorname{Im} z \cdot \bar{z}', z + z')$, where $z \cdot \bar{z}' = \sum z_j \bar{z}_j'$. We write $f * g(u) = \int_{\mathsf{H}^n} f(uv^{-1}) g(v)\, dv$, where dv is the Haar measure on H^n equalling Euclidean measure on $\mathsf{R} \times \mathsf{C}^n$. In the (h, u) coordinates, one thinks of U^{n+1} as $\mathsf{R}^+ \times \mathsf{H}^n$. The reason: if $u \in \mathsf{H}^n$, the "left translation" $T_u : \mathsf{U}^{n+1} \to \mathsf{U}^{n+1}$ by $T_u(h, v) = (h, uv)$ is then easily seen to be a holomorphic automorphism of U^{n+1}. Thus one thinks of H^n as $\partial \mathsf{U}^{n+1}$. One also has dilations: if $r > 0$, $D_r : \mathsf{H}^n \to \mathsf{H}^n$ by

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