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THE NON-VANISHING OF CERTAIN HECKE L-FUNCTIONS AT THE CENTER OF THE CRITICAL STRIP

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Introduction. The Hecke characters χ which will interest us in this paper are algebraic Hecke characters of weight one associated to imaginary quadratic fields. After an appropriate normalization which will be taken for granted, the meaning of "weight one" amounts to this: If $\mathfrak{n}$ is the conductor of χ and $\alpha \equiv 1 \bmod \mathfrak{n}$, then $\chi(\alpha\mathcal{O}) = \alpha$; here $\mathcal{O}$ denotes the ring of integers of the field K in question. There is one major restriction which we shall impose on χ in this paper: The conductor $\mathfrak{n}$ of χ must be relatively prime to $2\mathcal{O}$ and must divide $\sqrt{-D}\,\mathcal{O}$, where $-D < -4$ is the discriminant of K. We shall also assume that the values of χ on principal ideals lie in K. We let N be the positive integer such that $\mathfrak{n} \cap \mathbf{Z} = N\mathbf{Z}$.

For a Hecke character χ as above we let $L(s,\chi)$ denote the Hecke L-function associated to χ and put $\Lambda(s,\chi) = (DN)^{s/2}(2\pi)^{-s}\Gamma(s)L(s,\chi)$. The functional equation of $L(s,\chi)$ is $\Lambda(s,\chi) = W(\chi)\Lambda(2-s,\bar{\chi})$, where $W(\chi)$ is a complex number of absolute value one. Thus the center of the critical strip occurs at $s = 1$. If it happens that the functions $L(s,\chi)$ and $L(s,\bar{\chi})$ are equal, then $W(\chi)$ is either 1 or -1, and in the latter case, $L(1,\chi) = 0$.

If $-D \equiv 5 \bmod 8$ and $N = D$, then a calculation of $W(\chi)$ (in this connection see Gross [6]) shows that $L(1,\chi) = 0$. The purpose of this paper is to prove that the converse holds, apart from at most a finite number of exceptions.

THEOREM. *For all but finitely many* χ, *we have*: *If* $L(1,\chi) = 0$ *then* $-D \equiv 5 \bmod 8$ *and* $N = D$.

The proof of the theorem rests on work of Shimura ([11], [12]) concerning the special values of L-series associated to modular forms. Using Shimura's results, we prove the following: There is a function $G_{1,\epsilon}$ on the complex upper half plane with the property that $L(1,\chi) = 0$ if and only if $G_{1,\epsilon}$ vanishes at points on the upper half plane corresponding to ideal classes of K. (Here ϵ is the quadratic character associated to χ; see §1. The function $G_{1,\epsilon}$ is a holomorphic Eisenstein series of weight one for $\Gamma_0(N)$, but the proof makes no explicit use of this fact.) Suitable estimates of $G_{1,\epsilon}$ give necessary conditions on N and D for $G_{1,\epsilon}$ to vanish, and these conditions lead to the theorem. The approach is reminiscent of a technique of Deuring [5] and Chowla–Selberg [3].

There is an obvious defect in the theorem, which arises from our reliance on

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