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A NON-COMPACT VERSION OF PLÜCKER'S SECOND EQUATION

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Introduction. In this paper we shall prove a theorem and raise some questions related to non-compact algebraic geometry in $\mathbf{CP}^2$ (Cornalba and Griffiths [1]). An algebraic curve in $\mathbf{CP}^2$ can be considered as a holomorphic map $f: M \to \mathbf{CP}^2$, where M is a compact Riemann surface. Questions in non-compact algebraic geometry arise when one considers instead a mapping from a non-compact Riemann surface, for example $\mathbf{C}$. In particular, one would like to generalize classical results in algebraic geometry.

Let us consider the two classical Plücker equations

$$-2g + 2 + d^* + k = 2d \tag{1}$$

$$d(d-1) = d^* + 3k + 2\delta \tag{2}$$

where g is the genus, d the order, d^* the class (order of the dual), k the number of cusps, and δ is the number of double points. Classically, it is assumed that the singularities are of the simplest possible type, sometimes called traditional singularities (Griffiths and Harris [4]). The equations (1) and (2) may also be applied to the dual if it also has traditional singularities. The equation (1) has a non-compact version due to H. and J. Weyl [7] as follows: Suppose $f: \mathbf{C} \to \mathbf{C}^3 - \{0\}$ gives a holomorphic curve in $\mathbf{CP}^2$. Using the notation of Cowen and Griffiths [3], we have

$$-2\int_{|z| \leqslant r} d\,d^c \log|f|^2 + \int_{|z| \leqslant r} d\,d^c \log|f \wedge f'|^2 + n_1(r) = \int_{|z| = r} d^c \log \frac{|f \wedge f'|^2}{|f|^2} \tag{3}$$

where $d^c = (i/4\pi)(\bar{\partial} - \partial)$, and n_1 counts the ramification divisor. In the compact case, the integral on the left becomes $d^* - 2d + k$ and the integral on the right becomes $2g - 2$, and we get (1). If (3) is multiplied by dr/r and integrated, we get

$$-2T_0(r) + T_1(r) + N_1(r) = \frac{1}{4\pi} \int_{|z| = r} \log \frac{|f \wedge f'|^2}{|f|^4}$$

where T_0 and T_1 are the Nevanlinna growth functions for f and its dual. Now the right hand side may be estimated in terms of $\log T_0(r)$ using the lemma of the logarithmic derivative. The result is called the second fundamental theorem of Nevanlinna theory.

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