Vol. 47, No. 1 DUKE MATHEMATICAL JOURNAL © March 1980

NOTES ON HOMOLOGICAL ALGEBRA AND REPRESENTATIONS OF LIE ALGEBRAS

T. J. ENRIGHT AND N. R. WALLACH

§1. Introduction. In these notes several aspects of the interplay between homological algebra and representation theory for Lie algebras are studied.

In §2 derived functors are defined and some of their basic properties are discussed. In §3 the connection with representation theory is described. To begin we let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field k of characteristic zero and let $\mathfrak{h}$ and $\mathfrak{k}$ be subalgebras $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$ where $\mathfrak{h}$ is reductive and reductive in $\mathfrak{k}$ and $\mathfrak{g}$ and $\mathfrak{k}$ is reductive and reductive in $\mathfrak{g}$. For any Lie algebras $\mathfrak{a}, \mathfrak{b}$ with $\mathfrak{a} \subset \mathfrak{b}$, let $\mathcal{C}(\mathfrak{b}, \mathfrak{a})$ denote the category of all $\mathfrak{b}$-modules which are $U(\mathfrak{a})$-locally finite and semisimple as $\mathfrak{a}$-modules. Let S be the additive covariant functor defined on $\mathcal{C}(\mathfrak{g}, \mathfrak{h})$ by letting $S(X)$ be the maximal $U(\mathfrak{k})$-locally finite $\mathfrak{k}$-semisimple $\mathfrak{g}$-submodule of X, X an object in $\mathcal{C}(\mathfrak{g}, \mathfrak{h})$. The functor S along with its derived functors was introduced by G. Zuckerman in a series of lectures given at the Institute for Advanced Study (Spring 1978). In §3 we include some of the basic properties of S and its derived functors. Among these properties Lemma 3.4 and Proposition 3.7 appear to be new.

Zuckerman established in his lectures at IAS (Spring 1978) the importance of a duality theorem in the context of representation theory for semisimple Lie algebras (e.g., in connection with a construction of the discrete series modules [7]). The first main result of this article is a proof of this duality theorem (Theorem 4.3). A weaker duality theorem, essentially equivalent to Proposition 4.2, was proved by Zuckerman in his lectures at IAS. For $X \in \mathcal{C}(\mathfrak{k}, \mathfrak{h})$ let $X^{\sim}$ (resp. $X^{\approx}$) denote the maximal $\mathfrak{k}$-submodule of the algebraic dual X^* which is $U(\mathfrak{h})$ (resp. $U(\mathfrak{k})$) locally finite and semisimple as an $\mathfrak{h}$ (resp. $\mathfrak{k}$) module. If $R_{\mathfrak{g}}^{i}S$ denotes the ith right derived functor of S on $\mathcal{C}(\mathfrak{g}, \mathfrak{h})$, and if $m = \dim(\mathfrak{k}/\mathfrak{h})$ then the duality theorem (Theorem 4.3) asserts that for $X \in \mathcal{C}(\mathfrak{g}, \mathfrak{h})$ with finite dimensional $\mathfrak{h}$-isotypic subspaces, $R_{\mathfrak{g}}^{i}S(X)$ and $R_{\mathfrak{g}}^{m-i}S(X^{\sim})^{\approx}$ are naturally isomorphic as $\mathfrak{g}$-modules.

Now assume that $\mathfrak{k}$ and $\mathfrak{h}$ are reductive and $\mathfrak{t} \subset \mathfrak{h}$ is a Cartan subalgebra for both $\mathfrak{h}$ and $\mathfrak{k}$. Let P be a positive system for the roots Δ of $(\mathfrak{k}, \mathfrak{t})$ such that if Δ_1 denotes the roots of $(\mathfrak{h}, \mathfrak{t})$ and $P_1 = P \cap \Delta$ then $\alpha \in P_1$ is P_1-simple if and only if α is P-simple. Let $\mathcal{O}$ denote the category $\mathcal{O}$ introduced in [6] for the data $(\mathfrak{k}, \mathfrak{h}, \mathfrak{t}, P)$. The objects of the category $\mathcal{O}$ are the $\mathfrak{g}$-modules M having the following properties: (i) M is a finitely generated $\mathfrak{g}$-module, (ii) M is $U(\mathfrak{h})$-locally finite and semisimple as an $\mathfrak{h}$-module, (iii) if $\mathfrak{n} = \sum_{\alpha \in P} \mathfrak{g}_\alpha$ then M is $U(\mathfrak{n})$-locally

Received October 22, 1979.