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A MORSE INDEX THEOREM FOR NULL GEODESICS

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Let (M, g) be an arbitrary space-time of dimension $\geqslant 2$. This means that M is a smooth manifold with a countable basis, a smooth Lorentz metric g of signature $(-, +, \ldots, +)$, and a time orientation. We will denote the Lorentzian metric g by $\langle\ ,\ \rangle$ in the formulas for the index form below.

In Uhlenbeck [8, Theorem 1] a Morse theory for piecewise smooth timelike curves joining two chronologically related points in a globally hyperbolic space-time was given. Also Uhlenbeck was able to relate the homotopy of the loop space of the manifold itself to conjugate points of null geodesics in globally hyperbolic space-times satisfying certain metric growth conditions. Uhlenbeck [8, Theorem 4.8] obtained these results by constructing a Morse theory for nonspacelike curves in an arbitrary globally hyperbolic space-time as follows. Choosing a globally hyperbolic splitting $M = S \times (a, b)$, Uhlenbeck projected curves $\beta(t) = (c_1(t), c_2(t))$ in M onto the second factor (a, b), then showed that the functional $J(\beta) = \int (c_2'(t))^2\, dt$ yielded an index theory. From the construction used, however, this approach is limited to globally hyperbolic space-times or product manifolds.

Everson and Talbot [5, Theorem 4.6], on the other hand, worked directly with the usual Lorentzian energy functional to establish an index form inequality valid simultaneously for null and timelike geodesics in 4-dimensional globally hyperbolic space-times. Namely, the index of any nonspacelike geodesic was shown to be greater than or equal to the number of conjugate points (counting multiplicities) along the geodesic. The approach of Everson and Talbot would seem, however, to be restricted to globally hyperbolic space-times. They use a result of Clarke [3, p. 426], that any globally hyperbolic space-time may be C^k-embedded in Minkowski space of appropriate dimension, in order to treat the space of causal curves from the point of view of global analysis and Hilbert manifolds.

For an arbitrary space-time (M, g) of dimension $\geqslant 2$ a timelike Morse index theorem has previously been established, cf. Woodhouse [9, p. 145], Beem and Ehrlich [1, section 7]. Explicitly, if $c : [a, b] \to M$ is any timelike geodesic segment, then

$$\operatorname{Ind}(c) = \sum_{t \in (a, b)} \dim J_t(c)$$

where $J_t(c)$ denotes the $\mathbf{R}$-vector space of smooth Jacobi fields Y along c with $Y(a) = Y(t) = 0$. The purpose of this note is to indicate how a similar result may

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