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RELATIVE LIE ALGEBRA COHOMOLOGY AND UNITARY REPRESENTATIONS OF COMPLEX LIE GROUPS

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§1. **Introduction and Summary.** For reductive Lie groups the determination of the unitary dual of the group remains an open and elusive problem. In this article we solve a small part of this problem. This is done by first restricting the class of groups to complex semisimple Lie groups and then restricting the class of representations to those which might give rise to nonzero relative Lie algebra cohomology.

Let G be a connected simply connected complex Lie group with maximal compact subgroup K. Let $\mathfrak{g}_0$ (resp. $\mathfrak{k}_0$) denote the Lie algebra of G (resp. K) and let $\mathfrak{g}$ (resp. $\mathfrak{k}$) denote the complexification of $\mathfrak{g}_0$ (resp. $\mathfrak{k}_0$). If F is an irreducible finite dimensional $\mathfrak{g}$-module and V is an irreducible admissible $(\mathfrak{g}, \mathfrak{k})$-module then we let $H^*(\mathfrak{g}, \mathfrak{k}; F \otimes V)$ denote the relative Lie algebra cohomology groups as defined in [3] Chapter I. If $H^*(\mathfrak{g}, \mathfrak{k}; F \otimes V)$ is not zero then by Wigner's Lemma (cf. [3] I 4.2), V and the contragradient module to F have the same infinitesimal character. Therefore, in this case, V must have an infinitesimal character parameterized by the Weyl group orbit of a regular integral element. We say V has regular integral infinitesimal character. The main result of this article is:

THEOREM 6.1. *Let π be an irreducible unitary representation of G with regular integral infinitesimal character. Then there exists a complex parabolic subgroup Q of G and a one dimensional unitary representation σ of Q such that π is equivalent to the representation of G unitarily induced from Q and σ to G.*

By combining Theorem 6.1 and a version of Shapiro's Lemma due to P. Delorme [4] we obtain explicit formulae for $H^*(\mathfrak{g}, \mathfrak{k}; F \otimes V)$ (see Theorem 7.1). These formulae lead to the following vanishing theorem:

THEOREM 7.2. *Assume G is simple and assume V is a nontrivial irreducible unitary $(\mathfrak{g}, \mathfrak{k})$-module and F is a finite dimensional $\mathfrak{g}$-module. Then $H^r(\mathfrak{g}, \mathfrak{k}; F \otimes V) = 0$ for all integers $r < r_G$ where r_G is given in Table 1.*

For general real reductive groups Borel and Wallach [3] and Zuckerman [16] have proved the vanishing theorem for all r less than the split rank of G which,

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