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UNIQUENESS OF DELOOPING MACHINES

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In [12], J. P. May and the author showed that any two machines building spectra out of the usual kind of data must produce equivalent spectra from the same data. In this paper I prove a similar result for machines which deloop once the appropriate data. The theme of comparing two machines by constructing a third which contains parts of both is the same as in [12], but the way in which the third machine is built is radically different, as it must be. As in [12], it is possible to give axioms for a delooping machine that determine it uniquely up to homotopy equivalence. However, in this case it is not clear that every creditable delooping machine must satisfy the axioms. In fact, the only really difficult point in this paper is to show May's machine satisfies the axioms.

The real concrete content of this paper is the fact that for a topological monoid M (satisfying the usual cofibration condition that it has a non-degenerate basepoint), the delooping B_1M resulting from the application of May's machinery ([11; 13.1]) is homotopy equivalent to the usual classifying space BM. This important consistency statement has been unknown until now.

Roughly a creditable delooping machine should be a functor B from some category of "A_∞ data" to the category of connected based topological spaces. There are various creditable choices for what "A_∞ data" should mean, but certainly the category of topological monoids should be a subcategory of the category of A_∞ data. Also, an A_∞ datum should have an underlying topological space, and should provide it with the structure of a homotopy associative H-space. Secondly, a machine should come with a way to endow the loop space ΩX of each space X with the structure of an A_∞ datum. More precisely, there should be a functor "Ω" from the category of based spaces to that of A_∞ data, such that the underlying space of ΩX is naturally homotopy equivalent to the loop space of X. A good choice of Ω would be the functor sending each based space to the monoid of its Moore loops, but there are other possible choices of Ω not obviously equivalent to this (Definition 2.1, remark after Definition 3.6), and we wish to allow these possibilities. Finally, the functor B and its associated choice of Ω should be roughly inverse to each other: for Y a based connected space there should be a natural homotopy equivalence between Y and $B\Omega Y$; and for X an A_∞ datum such that π_0X is a group, there should be a natural homotopy equivalence of A_∞ data between X and ΩBX.

Given two such machines, B and B', one could hope to argue that for any A_∞ datum X, $B'X$ is naturally homotopy equivalent to BX, by observing the chain

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