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SURFACES IN $\mathbf{R}P^3$

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Let E be a manifold, M a submanifold, and $\mathcal{G}$ a group of diffeomorphisms of E. We want to study local differential geometric properties of M which are unchanged when M is replaced by gM, $g \in \mathcal{G}$. One approach is to let M be given by some defining equation, and then use the motions of $\mathcal{G}$ to simplify this equation. If this equation can be brought into a unique normalized form, then the coefficients in the normal form provide, more or less explicitly, all the information about the geometry of M. But a more common occurrence is that the equation cannot be completely normalized and there exists for M a family of normal forms parametrized by some subgroup of $\mathcal{G}$. We study such an example here. The technique we introduce for deriving invariants from a knowledge of normal forms should be applicable to other geometries.

Let $E = \mathbf{R}P^3$, M a two-dimensional submanifold, and $\mathcal{G} = GL(4, \mathbf{R})/\{\text{center}\} \equiv PGL(4, \mathbf{R})$. The action of $\mathcal{G}$ on E is given by identifying E with the set of lines in $\mathbf{R}^4$. The study of local properties of $M \subset E$ invariant under $\mathcal{G}$ is equivalent to the study of local properties of $M \subset \mathbf{R}^3$ invariant under the group of projective transformations. M is said to be negatively curved if for each point $p \in M$ the tangent plane to M at p intersects M near p in two distinct curves. If M is negatively curved, then so is gM, for each $g \in \mathcal{G}$, since $\mathcal{G}$ maps planes to planes. Such a manifold admits a system of asymptotic coordinates. These coordinates are used to realize the normal forms. The same procedure may work for positively curved manifolds upon considering complex asymptotic coordinates.

Another paper by the author which was motivated by similar questions is [3]. It might be useful to contrast these two geometries. The geometry of strictly pseudoconvex real hypersurfaces in $\mathbf{C}^{n+1}$ is modelled on that of the hyperquadric $Q_1 = \{(z_1, \ldots, z_{n+1}) \mid \operatorname{Im} z_{n+1} = \sum_1^n |z_i|^2\}$ which is considered as a homogeneous space $SU(n+1, 1)/H_1$. The motions in the geometry are given by the pseudogroup of local biholomorphisms. The geometry of negatively curved surfaces in $\mathbf{R}P^3$ is modelled on that of $Q_2 = \{(x, y, z) \mid z = xy\}$ which may be considered as a homogeneous space $SO(2, 2)/H_2$. The motions in the geometry are given by the group $PGL(4, \mathbf{R})$. (A more natural comparison might be to replace Q_1 by $\{(z_1, \ldots, z_{n+1} \mid \operatorname{Im} z_{n+1} = -|z_1|^2 + |z_2|^2 + \ldots + |z_n|^2\} = SU(n, 2)/H$ and do without pseudoconvexity. The discussion below would apply equally well to this geometry.)

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