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CURVATURE ON THE FERMAT CUBIC

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Consider the particular nonsingular algebraic curve $C \subset \mathbb{CP}^2$, the complex projective plane, defined by the equation Fermat made famous:

$$F(X, Y, Z) = X^3 + Y^3 + Z^3 = 0. \tag{1}$$

This is an elliptic curve. We will view C as being endowed with the metric induced by the Fubini-Study metric on $\mathbb{CP}^2$. Let $K : C \to \mathbb{R}$ denote the Gaussian curvature function on C determined by this metric.

In this paper we show that K is a Morse function. We explicitly determine the critical points and give two characterizations of them. We show that the critical points are the points left fixed by some projective linear automorphism on C. We also show, with respect to any group structure on C, in which 0 is a flex, that the critical points are the union of the points of order three (the flexes), the six-torsion points, and the points of order nine, whose triple is on the same axis as the flex 0.

The organization of the paper is as follows. In section one, we review the Fubini-Study metric and recall the curvature formula. In section two we discuss the group structures on C and the group of projective linear automorphisms of C. In section three, we determine points of C left fixed by some automorphism and show that each of these is a critical point. In section four, we give the quadratic approximation of K at these critical points. The last section is devoted to the statement and proof of the main result.

§1. The Fubini-Study metric and the curvature formula

The Fubini-Study metric [1] is a Kähler metric on $\mathbb{CP}^2$, which is invariant under transformations of $\mathbb{CP}^2$, induced by unitary transformations of $\mathbb{C}^3$. Up to multiplication by a constant, it is the unique such metric. We normalize so that the curvature of a projective line is 2.

The Fubini-Study metric is also geometrically natural in the following sense. Let $\mathbb{C}^3 \to \mathbb{CP}^2$ denote the usual quotient map. Suppose $Z(t_1, t_2)$ is a local holomorphic lifting of some open set in $\mathbb{CP}^2$. Then the metric is given by

$$\frac{|Z \wedge dZ|^2}{\|Z\|^4}$$

where $\| \ \|$ denotes the usual Hermitian norm in $\mathbb{C}^3$.

Suppose now that $G(X, Y, Z)$ is a homogeneous polynomial of degree d with $P \in G = 0$, a nonsingular point. Assume $G = 0$ is endowed with the induced

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