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THE RELATIONSHIP BETWEEN THE INDEX OF A STRONGLY STABLE PERIODIC LINEAR HAMILTONIAN VECTORFIELD AND THE INDEX OF ITS STABILITY DOMAIN

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§1. Introduction.

Let (V, σ) be a $2n$ dimensional vectorspace over $\mathbb{R}$ with symplectic form σ and let $\mathrm{Sp}(\sigma, \mathbb{R})$ be the Lie group of all linear maps of V into itself which preserve σ. Suppose that $\Phi : [0, 1] \to \mathrm{Sp}(\sigma, \mathbb{R})$ is a continuous curve such that $\Phi(0) = I$ and $\Phi(1)$ is *strongly stable*, that is, there is an open neighborhood of $\Phi(1)$ in $\mathrm{Sp}(\sigma, \mathbb{R})$ consisting of semisimple mappings whose eigenvalues lie on the unit circle. Then $\Phi(1)$ is in the image of the Lie algebra $\mathrm{sp}(\sigma, \mathbb{R})$ of $\mathrm{Sp}(\sigma, \mathbb{R})$ under the exponential mapping. We write $\Phi(1) = \exp A$ with $A \in \mathrm{sp}(\sigma, \mathbb{R})$ as close as possible to 0. Laying the curves Φ and $\varphi : [0. 1] \to \mathrm{Sp}(\sigma, \mathbb{R}) : t \to \exp(1 - t)A$ end to end gives a closed curve γ in $\mathrm{Sp}(\sigma, \mathbb{R})$ whose homotopy class $[\gamma] \in \pi_1(\mathrm{Sp}(\sigma, \mathbb{R})) \simeq \mathbb{Z}$ is called the *index* $\mathscr{I}(\Phi)$ *of the strong stability domain* of Φ [3, p. 169].

On $V \times V$ define a symplectic form τ by $\tau((v_1, w_1), (v_2, w_2)) = \sigma(v_1, v_2) - \sigma(w_1, w_2)$. Let $\Lambda(V \times V)$ be the Lagrangian Grassmannian, that is, the manifold of all Lagrangian subspaces of $(V \times V, \tau)$ [1, p. 181]. Since $gr\ \Phi : [0, 1] \to \Lambda(V \times V) : t \to gr\ \Phi(t)$-graph of $\Phi(t)$ is a continuous curve of Lagrange spaces, its index ind $gr\ \Phi$ is defined [1, p. 181, equation 2.14] and is equal to the number of nonzero intersections of the curve with a fixed Lagrangian subspace μ plus a correction term involving $gr\ \Phi(0)$, $gr\ \Phi(1)$ and μ which makes the index independent of the choice of μ.

The purpose of this paper is to establish the following relation between the index of the strong stability domain of Φ and the index of the curve of Lagrange spaces $gr\ \Phi$:

1) $$\mathscr{I}(\Phi) = \frac{1}{2}\ (\mathrm{ind}\ gr\ \Phi + \mathrm{ind}\ \sigma^{\#} \circ A) - n$$

where $\sigma^{\#} \circ A : V \times V \to \mathbb{R} : v, w \to \sigma(Av, w)$ is a symmetric bilinear form because $A \in \mathrm{sp}(\sigma, \mathbb{R})$.

Suppose that Φ is the fundamental solution curve of a period 1 periodic linear Hamiltonian vectorfield X_H on (V, σ), that is, $\Phi : [0, 1] \to \mathrm{Sp}(\sigma, \mathbb{R}) : t \to \Theta_t$ where $\Theta : \mathbb{R} \times V \to V$ is the flow of X_H. If X_H is obtained by the Legendre transformation from a variational problem with periodic boundary conditions

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