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ON THE DISTRIBUTION OF FEKETE POINTS

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1. Introduction

Let $E \subset R^n$, $n \geq 3$, be a compact set and N a given positive integer. A system of points $P_1, \cdots, P_N \in E$ which minimizes $\sum\sum_{i \neq j} |P_i - P_j|^{2-n}$ is called a system of Fekete points of E. (Notice that this represents a stable equilibrium distribution of N equal point charges on E). The purpose of this note is to find estimates of the distance d from a Fekete point P_i to its closest neighbour P_i^*. Using complex methods, Kövari and Pommerenke [1] found that if $E \subset R^2$ is a sufficiently smooth curve then $C_1N^{-1} \leq d \leq C_2N^{-1}$. In the case when $n \geq 3$ and E is a closed $C^{1,\alpha}$ surface, Sjögren [3] found the estimate $d \leq C\, N^{-\gamma}$, where $\gamma = \frac{1}{2}(n-1)^{-2}$. We can show the following estimate:

THEOREM. *Let $S \subset R^n$, $n \geq 3$, be a closed, compact $C^{1,\alpha}$-surface, where $0 < \alpha < 1$, that separates R^n into two components. Then there are positive numbers $C_i = C_i(S)$, $i = 1, 2$, such that if N is a positive integer and $P_1, \cdots, P_N$ is a system of Fekete points of S then*

$$C_1 r_N \leq |P_i - P_i^*| \leq C_2 r_N,\ 1 \leq i \leq N, \tag{1.1}$$

where $r_N = N^{-1/(n-1)}$.

2. The main result

We start by recalling that a $C^{1,\alpha}$-surface in R^n is a closed, bounded $(n-1)$-dimensional surface S such that S can be covered by finitely many open right circular cylinders whose bases have a positive distance to S and to each cylinder C there is an orthonormal coordinate system (x, y), $x \in R^{n-1}$, $y \in R$, such that the y-axis is parallel to the axis of symmetry of C and $C \cap S = C\{(x, y): y = \phi(x)\}$, where $\phi: R^{n-1} \to R$ is a C^1-function such that $|\nabla\phi(x) - \nabla\phi(z)| \leq M|x - z|^\alpha$, where ∇ denotes the gradient.

We shall from now on assume that $S \subset R^n$, $n \geq 3$, is a $C^{1,\alpha}$-surface for some α, $0 < \alpha < 1$, such that S separates R^n into two components D and D_∞ where D_∞ denotes the unbounded one. We denote by dS the surface measure element on S and by λ the equilibrium measure of S, i.e., the unique positive measure on S with total mass 1 minimizing

$$\iint |P - Q|^{2-n} d\lambda(P) d\lambda(Q).$$

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