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VARIATIONAL PROPERTIES OF THE COMPLEX MONGE-AMPÈRE EQUATION I. DIRICHLET PRINCIPLE

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§1. Introduction

In this paper we study the Dirichlet Principle associated to the Dirichlet problem for the complex Monge-Ampère equation. The work is based on some integral formulas which are complex analogues of a formula of S. Bernstein. Applications of the Dirichlet principle are made to prove a general maximum principle for the complex Monge-Ampère operator (Theorem 4.11), to prove the existence of solutions of the Dirichlet problem on 2-dimensional Stein manifolds (Theorem 4.14), and to prove a trace theorem for plurisubharmonic functions.

The Dirichlet Principle studied is connected with the Dirichlet problem

$$u \text{ plurisubharmonic on } \Omega$$

$$(1.1) \qquad (dd^c u)^n = dd^c u \wedge \cdots \wedge dd^c u = f \geq 0 \text{ on } \Omega$$

$$u = \phi \text{ on } \partial\Omega$$

for a bounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ with smooth boundary. For $u \in C^2(\Omega)$ one may compute

$$(dd^c u)^n = 4^n n! \det \left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right) dV$$

where dV is the usual volume form on $\mathbb{C}^n$. If $P(\Omega)$ denotes the cone of plurisubharmonic functions on Ω, then $(dd^c u)^n$ is positive whenever $u \in C^2(\Omega) \cap P(\Omega)$. It has been shown (see [2]) that $(dd^c)^n : C^2(\Omega) \cap P(\Omega) \to M(\Omega)$ has a continuous extension to a map $C(\Omega) \cap P(\Omega) \to M(\Omega)$, where $M(\Omega)$ is the set of nonnegative Borel measures with the weak topology. The operator $(dd^c)^2$ also has a continuous extension to $L_1^2(\Omega) \cap P(\Omega)$, (Proposition 3.6). By $(dd^c)^n$ we will mean one of these extensions, and the solutions to the problem (1.1) are taken in this generalized sense.

Let us recall the formula of S. Bernstein

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