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LOWER BOUNDS FOR EIGENVALUES OF REGULAR STURM-LIOUVILLE OPERATORS AND THE LOGARITHMIC SOBOLEV INEQUALITY

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The one dimensional logarithmic Sobolev inequality states that

$$\int_{-\infty}^{\infty} |f(x)|^2 ln|f(x)|d\nu(x) \leq \int_{-\infty}^{\infty} |f'(x)|^2 d\nu(x) + \|f\|_2^2 ln\|f\|_2$$

where ν is normalized Gauss measure,

$$d\nu(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \text{ and } \|f\|_2$$

denotes the $L^2(\nu)$ norm of f. Nelson [3] proved an equivalent inequality using ideas from quantum field theory; Gross [2] showed the equivalence and gave an elegant proof using the central limit theorem.

In an attempt to better understand and generalize the inequality, I discovered that it is a consequence of, and essentially equivalent to, a new and quite elementary estimate from below for the first eigenvalue of regular Sturm-Liouville operators on finite intervals. As a matter of fact this estimate can be obtained for sufficiently well behaved domains in $\mathbb{R}^n$, not a priori, but with the use of the n dimensional logarithmic Sobolev inequality. This topic we plan to pursue in a subsequent paper. A novel feature of our method, which we do explore in this paper, and which appears to be effective in the one dimensional setting only, is the ease with which it gives good estimates from below for all the eigenvalues.

The first portion of this paper will be devoted to Sturm-Liouville operators on finite intervals. The second portion will give the connections with the logarithmic Sobolev inequality, and show how analogous inequalities may be obtained.

At the expense of some generality, we are taking a very elementary point of view. Thus all our integrals are Riemann integrals, or improper Riemann integrals of positive functions, possibly taking the value plus infinity.

Definition. By $C^1(a, b)$ (respectively $C^2(a, b)$), we denote the space of functions continuous on the closed interval $[a, b]$, vanishing at the end points, and with one (respectively two) continuous derivatives on the interior.

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