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TOTALLY GEODESIC SUBMANIFOLDS OF SYMMETRIC SPACES, I

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§1. Introduction.

An isometric immersion $f: M \rightarrow N$ of a Riemannian manifold M into another Riemannian manifold N is called *totally geodesic* if the geodesics in M are carried into geodesics in N. In this paper we will completely classify all the complete, connected, totally geodesic submanifolds of the complex quadratic hypersurfaces: $Q_m = SO(m + 2)/SO(2) \times SO(m)$, $m > 1$.

Our main result is, in a simplified version, that every maximal connected totally geodesic submanifold M of Q_m is a connected component of the fixed point set of some involutive isometries of Q_m. More concretely M is one of the following three spaces: (1) Q_{m-1}, (2) the Riemannian product $S^p \times S^q$, $m = p + q$, of two spheres of the same curvature, and (3) the complex projective space of complex dimension n (if $m = 2n$). Their immersions as totally geodesic submanifolds in Q_m are unique up to an isometry of Q_m. Furthermore we shall study more details of these spaces. Among these three spaces above, Q_{m-1} is the only complex submanifold of Q_m, the immersion of $S^p \times S^q$ is totally real, and the complex projective space is neither a complex submanifold nor a totally real submanifold of Q_m. If M is a non-maximal, complete, connected, totally geodesic submanifold of Q_m, then M is contained in Q_{m-1} in some position, except for the real projective space of half dimension which is contained in the complex projective space mentioned above for an even m. Consequently, every connected totally geodesic submanifold of dimension > 1 of Q_m is an open submanifold of the common fixed point set of a finite number of involutive isometries of Q_m.

The choice of the space Q_m was made out of our current interests beside its simplicity. We do not think however that the feasibility is limited to Q_m but we hope that our method will work for other symmetric spaces. In section 2 we will give examples of totally geodesic submanifolds of Q_m with detailed descriptions. Our main results will be proved in section 3. In the last section we will give some topological descriptions of those totally geodesic submanifolds. In particular, we will prove that each homology group $H_k(Q_m; Z)$, $k < 2m$, is spanned by the classes of totally geodesic submanifolds. And there is a maximal connected totally geodesic submanifold M of Q_m such that the differentiable manifold Q_m is the union of the normal bundles to M and to its focal manifold with nonzero vectors identified in some way.

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